The ADHM Construction and its applications to Donaldson Theory

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A thesis submitted to the University of Warwick
for the degree of Philosophiae Doctor

Mathematics Institute
University of Warwick
July 2001

κατοικῆσαι τὸν Χριστὸν διὰ τῆς πίστεως ἐν ταῖς καρδίαις ὑμῶν· ἐν ἀγάπῃ ἐρριζωμένοι καὶ τεθεμελιωμένοι ἵνα ἐξισχύσητε καταλαβέσθαι σὺν πᾶσι τοῖς ἁγίοις τί τὸ πλάτος καὶ μῆκος καὶ ὕψος καὶ βάθος, γνῶναί τε τὴν ὑπερβάλλουσαν τῆς γνώσως ἀγάπην τοῦ Χριστοῦ, ἵνα πληρωθῆτε εἰς πᾶντὸ πλήρωμα τοῦ θεοῦ.

ΕΠΙΣΤΟΛΗ ΠΡΟΣ ΕΦΕΣΙΟΥΣ .

That Christ may dwell in your hearts by faith; that ye, being rooted and grounded in love, may be able to comprehend with all saints what is the breadth, and length, and depth, and height; and to know the love of Christ, which passeth knowledge, that ye may be filled with all the fulness of God.

The letter to the Ephesians 3:17-19.

Contents

Acknowledgements

I have many thanks to give:

- First, to my supervisor Victor Pidstrigatch, for his support, kindness, seemingly unlimited patience; a true gauge theoretical thesaurus whose good humour lightened my otherwise heavy load.
- To my "other" supervisor, Mario Micallef, for his willingness to help, his unconditional support, sound advice and robust good humour even at unearthly hours in the morning.
- To my friends, Ali Craw, the Andrews (Stacey and Clow), Dan Ryder, Robert Reid, Alan and John Carey, Paul Cooper, Rita Gaio, Diego Matessi, Eivind Eriksen and Elisabeta Nedita, all of whom gave me companionship and detailed information to get by.
- To my departmental mentor, David Mond, for being someone to turn to in moments of singular difficulty.
- To the other Geometers, Miles Reid, John Rawnsley and John Jones, the Topologists Brian Sanderson and Colin Rourke, the Analyst David Elworthy, and to the Algebraists Bruce Westbury and Charu Harjarnavis for their many and welcome ideas, and for having the patience with my persistent badgering.
- To the mathematicians on the outside: Frances Kirwan, Hiraku Nakajima, Werner Nahm, Andrew Dancer, Edward Corrigan and my erstwhile tutor Dietmar Salamon who supplied me with papers, ideas and perceptions.
- To Lara Alcock for putting up with my "desk of Damocles" and other shenanigans in the office.
- To Pauline, Joan and Irene, for plentiful supplies of Tea, Coffee, Jaffa Cakes and friendship.
- To Ros Barber, for her kindness, warmth, and infallible knowledge of where things are in the Library.
- To the "Powers That Be" that permitted me to lecture Lie Groups.
- To Ysette Weiss-Pidstrigatch for her friendship, and for kindly sorting me out in Germany.
- To Gill Shipp, for her tolerance, kindness, assistance, support, education, advice, encouragement, guidance and the use of her home for 7 years.
- To David Embery for being David Embery.

- To Graham and Pam for starting me off.
- To Hazel St. John, Ailsa Mackintosh, Comfort Davis and Margaret Oliver.
- To Trevor Hawkes, Alyson Stibbard, and Sandra Cook.
- To Jack Cohen and Ian Stewart for their stimulating thoughts, and for the fish, fire-eels, the plec and Dougal mark 2.
- To all the members of St Stephen the Martyr's congregation for putting up with my abominable organ playing.
- To the rectors, vicars, pastors and chaplains whom I have encountered while I've been here: Rev. Peter Smith, Jackie Gare, Rev. Clive Gregory, Rev. David Scott, His Grace, Rt. Rev. Philip Pargeter, The Venerable Peter Lock, Father John Hambidge and Rev. Mark Bratton, without any of whom, my life would be darker, and my thoughts duller.Oh, and of course the Canley Evangelical Crew.
- To my family: Eileen and Malcolm Munn, Matthew and Judith Arthur and latterly Louise Elizabeth Arthur, for all their love, support, encouragement and patient endurance with a seemingly eternal student.
- and most of all to the One True Sovereign Lord God, my Creator, my Light, my Life, my Inspiration, and my Salvation, for all those things that I cannot adequately put into words or list exhaustively.

Each has played a vital rôle in this work, small or great, and to each I am eternally grateful.

Declaration

Chapter 1 is an elaboration of the paper [23] by Professor Werner Nahm using elements of the theory detailed in Donaldson and Kronheimer. My aim was to formalise that which Nahm set out in a framework more conducive to mathematicians. It thus follows the structure of the paper by Nahm, yet I have included more detail. Sections 1.1.1 and 1.1.2 are based upon a seminar that I gave more or less every year to the new postgraduates, and is thus a brief survey of the theory of Spin geometry.

Chapter 2 is, to the best of my knowledge, original unless otherwise stated.

Chapter 3 contains the theory of nonabelian localisation for symplectic manifolds by Professors Frances Kirwan and Lisa Jeffrey, which I have closely followed in proving a similar result for HyperKähler Manifolds. The results in basic equivariant cohomology were derived from the thesis of Doctor M. J. Selby

Chapter 4 is to the best of my knowledge original unless otherwise stated.

On the whole, to the best of my knowledge, the material in this thesis is original unless otherwise stated.

Introduction

In the 1980s, Simon Donaldson discovered a way to calculate invariants of smooth structures using the solutions of a particular partial differential equation, namely the solutions to anti-self dual (ASD) equations.

Given a compact 4-manifold X and a SU(2) vector bundle $\pi : E \longrightarrow X$ with

$$\langle c_2(E), [X] \rangle = k,$$

he looked at all connections A which satisfied the ASD equation

$$\star F(A) = -F(A)$$

or

$$F(A)^+ = 0.$$

Although this is an infinite dimensional space, it has many symmetries given by the gauge group $\mathcal{G}$ consisting of all bundle automorphisms of E. From work done by Sir Michael Atiyah, Nigel Hitchin, Isadore Singer, Gerard t'Hooft and many others, it was known that the moduli space $\mathcal{M}_k$, i.e. the space of all ASD connections modulo the gauge group $\mathcal{G}$, was finite dimensional. In particular, the dimension was given by

$$\dim \mathcal{M} = 8k - 3\left(1 - b_1(X) + b_2^+(X)\right)$$

using index theory in [7], where the b_is are the betti numbers of X and b_2^+ the number of positive eigenvalues of the intersection form on the second homology of X.

Donaldson had the idea of forming a universal bundle $\mathbb{E} \longrightarrow X \times \mathcal{M}$ by

1. pulling back the bundle E over the product $X \times \mathcal{A}$ where $\mathcal{A}$ is the space of all connections on E giving a $\mathcal{G}$-equivariant bundle

2. pushing this to a quotient over $X \times \mathcal{B}$, the space of all connections modulo $\mathcal{G}$

3. restricting the result to the moduli of ASD connections.

In fact, due to complications, he applied this technique to AdE rather than E to obtain a universal SO(3) bundle Ad$\mathbb{E}$ over $X \times \mathcal{M}$.

He reasoned that we now have a way to encode data on X into data on $\mathcal{M}$. Given $\Sigma \in H_q(X)$, we may find its Poincaré dual $\mathrm{PD}(\Sigma) \in H^{4-q}(X)$. By pulling this back to $X \times \mathcal{M}$ we obtain a class in $H^{4-q}(\mathcal{M})$

$$\mu(\Sigma) = \left\langle \frac{1}{4} p_1(\mathrm{Ad}\mathbb{E}) \smile \mathrm{PD}(\Sigma), [X] \right\rangle. \tag{1}$$

So given l elements $\Sigma_i \in H_{q_i}(X)$ such that

$$4l - \sum_{i=1}^{l} q_i = 8k - 3\left(1 - b_1(X) + b_2^+(X)\right)$$

we can take the product of all of their μ-classes and evaluate it over the fundamental class of $\mathcal{M}$, i.e. work out

$$\langle \mu(\Sigma_1) \smile \ldots \smile \mu(\Sigma_l), [\mathcal{M}] \rangle .$$

This gives us a polynomial associated to X in the elements of the homology of X and is an invariant of the smooth structure on X under diffeomorphism.

There are snags with this construction. First, that the manifold $\mathcal{M}$ is rarely compact; we have to make some kind of compactification in order to make sense of the fundamental class $[\mathcal{M}]$. The second problem is the nature of the calculation of these polynomials as we are dealing with infinite dimensional spaces and infinite dimensional groups.

In their pioneering book, The Geometry of 4-Manifolds [13], Donaldson and Peter Kronheimer detail how to find a canonical representative $c \in \Omega^4(X \times \mathcal{M})$ of the class $p_1(\mathrm{Ad}\mathbb{E})$, we detail this in Section 2.3. Theoretically, we should be able to find a representative of the Donaldson μ-map of a subspace Σ of X by calculating

$$\int_X c \wedge \mathrm{PD}(\Sigma).$$

Ed Witten speculated that this was a reliable way of calculating these polynomials.

However, the physicists also using the Donaldson polynomials as a way to understand their Topological Field Theory, tried this approach and found anomalies. The simplest of these anomalies was brought to my attention in the paper by Damiano Anselmi [1] in which he tries to calculate these polynomials using the method of representatives in the situation of $X = \mathrm{S}^4$ and $k = 1$. In this case, the moduli space $\mathcal{M}$ is contractible, hence all the μ classes and polynomials vanish. He took Σ to be a point and a cylinder $\Xi = \mathrm{S}^3 \times [0, \infty)$ in the moduli space and found that the integral of the μ class of Σ over Ξ came out to be non-zero if the point were in the interior of the sphere. He went on to show that there is some kind of linking theory within this approach that had not been foreseen and conjectured on a similar result for higher k. We demonstrate the reasons for the anomalies as a case of integrating over the wrong space.

The method that Anselmi used was that of BRST theory and he somehow incorporates the ADHM theory of instantons to detail a specific representative of the μ class.

The ADHM method of constructing instantons was a result of the twistor programme started by Penrose. In their famous paper [6], Atiyah and Hitchin in Britain and the Soviet Mathematicians Vladimir Drinfel'd and Yuri Manin adapted Jeffrey Horrocks' construction for building coherent sheaves to build connections out of matrix data. In Chapter 1, we detail this construction by expounding both on the algebraic geometric proof given by the mathematicians (in particular Kronheimer and Hiraku Nakajima, [20, 24, 25]) but removing the heavy dependence on spectral sequences by expounding the approach of the mathematical physicist Werner Nahm [23].

The beauty of this construction is that it establishes a 1-1 correspondence between instantons on S^4 with instantons on $\mathbb{R}^4$ with conditions on the asymptotics at ∞.It also uses finite dimensional data of matrices and so in theory a lot of the classical theorems should apply. However this is not always so; the data is too complicated to yield very many topological results about the moduli spaces.

Donaldson noticed that by considering the framed gauge group $\mathcal{G}_0$ consisting of all gauge transformations which are the identity at a given fixed point of S^4, the moduli space

$$\widetilde{M} = \frac{\{F(A)^+ = 0\}}{\mathcal{G}_0}$$

is the hyperKähler quotient by $\mathcal{G}_0$ of the quaternionic space $\mathcal{A}$ with moment map $\vec{\mu}$ [1] given by

$$\begin{gathered}\vec{\mu} : \mathcal{A} \longrightarrow \Omega^0(\mathbb{R}^4; \mathrm{Ad}E \otimes \Lambda^2_+ \mathrm{T}^*\mathbb{R}^4) \\ \vec{\mu}(A) = F(A)^+.\end{gathered}$$

It was also discovered that the ADHM construction gives a hyperKähler reduction diffeomorphic to $\widetilde{\mathcal{M}}$ and that there was some duality between the two situations which the physicists call T-duality.[2] This duality indicates that the representatives of the μ-class obtained by the infinite dimensional construction should be exactly the same at the level of forms as those obtained by the finite dimensional version. This work toward showing this is given in Chapter 2. Theoretically this should lead to an easier way to calculate and integrate these forms. However this is far from true. There is no obvious parameterisation for the moduli spaces except in the case $k = 1$ when the moduli space of instantons over $\mathbb{R}^4$ is the 5 dimensional upper-half space.

To rectify this we try to use equivariant cohomology, a tool used for calculating cohomological properties of symplectic quotients. We give the theory of equivariant integration in the symplectic case using the combined approaches by Lisa Jeffries and Frances Kirwan in [18], Victor Guillemin and Jaap Kalkman [14] and Ed Witten in [29]. We then try to adapt these arguments to the case of hyperKähler reduction in order to lift the calculation of integrals from the moduli space to the finite dimensional vector space of which the moduli space is the hyperKähler quotient. This takes place in Chapter 3.

[1] Yes, I'm afraid the standard notation for moment map coincides with the notation of the Donaldson map. I hope that the context will be clear whenever μ is used.

[2] "T" standing for "target space"

Our aim is then to use localisation formulæ to express the integrals over smaller spaces, in the hope of creating a recursive linking theory in the case of higher charges. However the result is not really enlightening as we had hoped, but we do have some comfort in the fact that this agrees with the infinite dimensional construction of Donaldson numbers and polynomials.

Chapter 1

Formalising Instanton Construction after Nahm

1.1 Preliminaries on Spin Geometry

1.1.1 Spinors on vector spaces

Let V be a real vector space of four-dimensions with an inner product $\langle \cdot, \cdot \rangle$. Then SO(4) acts on V as the group of isometries. Now there is a simply-connected double cover of SO(n) namely Spin(n). In four dimensions, we can make the following isomorphism,

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2).$$

Since the Lie group of special unitary transformations is involved here, we would like a way of comparing V with two 2-dimensional complex vector spaces with hermitian inner products on which each component of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ can act.

Let S be a two dimensional complex vector space with inner product and symplectic form $\omega \in {\Lambda_{\mathbb{C}}}^2 S$ which has unit determinant with respect to the metric. Now, SU(2) acts on S as the symmetry group preserving metric and symplectic form. On S, there an anti-linear endomorphism $J \in \mathrm{End(S)}$ with the property that

$$J^2 = -\mathbb{1}.$$

We can define this naturally using the symplectic form ω, i.e.

$$\langle x, Jy \rangle_S = \omega(x, y)$$

Note that if we were given such a J, we could equally define the form ω using the same equation.

The pair (S, J) has thus become a quaternionic space and so Sp(1) acts upon it. Now, Sp(1) consists just of the unit quaternions and acts on S by

$$(x_1 + x_2\mathbb{i} + x_3\mathbb{j} + x_4\mathbb{k})(v) = x_1 v + ix_2 v + x_3 Jv + ix_4 Jv \qquad \forall\, v \in S.$$

Each element of Sp(1) can be identified with an endomorphism of S. Using standard bases, $x_1 + x_2\mathbb{i} + x_3\mathbb{j} + x_4\mathbb{k} \in \mathrm{Sp}(1)$ corresponds with the transformation

$$\begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \qquad \in \mathrm{SU}(2).$$

Now consider two such vector spaces S_+ and S_- with inner products $\langle\cdot,\cdot\rangle_+$, $\langle\cdot,\cdot\rangle_-$ and symplectic forms ω_+ and ω_- respectively. Denote by J_+ and J_- their respective quaternionic structures and by $\mathrm{Hom}_J\,(S_+, S_-)$ the space of $A \in \mathrm{Hom}(S_+, S_-)$ such that

$$AJ_+ = J_- A.$$

$\mathrm{Hom}_J\,(S_+, S_-)$ consists of precisely the matrices of the above form and this is a four dimensional real vector space.

Conversely, given a four dimensional vector space V with inner product, a spin-structure on V is a pair of 2-dimensional hermitian vector spaces with symplectic form (S_+, S_-) such that

$$V \cong \mathrm{Hom}_J\,(S_+, S_-)\,.$$

Notice that

$$V \otimes \mathbb{C} \cong \mathrm{Hom}(S_+, S_-).$$

Now we have an linear isomorphism $\gamma\colon V \longrightarrow \mathrm{Hom}_J\,(S_+, S_-)$, so given an orthonormal basis of V $\{e_1, e_2, e_3, e_4\}$, set

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma(e_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$\gamma(e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma(e_4) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We can identify $\Lambda^2(V)$ with $\mathrm{End}(S_+)$ using γ and the facts that if $e, e' \in V$ are orthogonal unit vectors then

1. $\gamma(e)^*\gamma(e) = \mathbb{1}$,
2. $\gamma(e')^*\gamma(e) + \gamma(e)^*\gamma(e') = 0$.

So the map γ extends to $\Lambda^2(V) \mapsto \mathrm{End}(S_+)$ via

$$\gamma(e \wedge e')\varphi = -\gamma(e)^*\gamma(e')\varphi \qquad \forall \varphi \in S_+.$$

Now the Hodge $\star$-operator in four dimensions satisfies $\star\star = \mathbb{1}$ on $\Lambda^2(V)$, hence there is a decomposition of $\Lambda^2(V)$ into ± 1 eigenspaces

$$\Lambda^2(V) \cong \Lambda^2_+(V) \oplus \Lambda^2_-(V).$$

Under this decomposition we find that $\gamma(\Lambda^2_-(V)) = 0$ and hence

$$\gamma\colon \Lambda^2_+(V) \longrightarrow \mathrm{End}(S_+).$$

It can also be shown that this is actually an action of the imaginary quaternions. A choice of basis of V will give a specific representation of $\mathbb{i}$, $\mathbb{j}$ and $\mathbb{k}$, from a basis of $\Lambda^2_+(V)$.

Now we will write

$$a \cdot \psi \qquad \text{for} \qquad \gamma(a)\psi$$

whenever $a \in V$ and $\psi \in S_+$.

Using this, for each non-zero $\varphi \in S_+$ we get an isomorphism

$$\begin{array}{ccc} V & \longrightarrow & S_- \\ x & \mapsto & x \cdot \varphi. \end{array}$$

Since S_- is a complex space, we have a natural complex structure induced by φ on V, i.e. define $J_\varphi \in \mathrm{End}(V)$ by

$$(J_\varphi x) \cdot \varphi = i(x \cdot \varphi).$$

Proposition 1.1.1 *For any non-zero $\lambda \in \mathbb{C}$ and $\varphi \in S_+$, φ and $\lambda\varphi$ induce the same complex structure on V, i.e.*

$$J_{\lambda\varphi} = J_\varphi.$$

Proof
For any $x \in V$

$$\begin{aligned} (J_{\lambda\varphi} x) \cdot \lambda\varphi &= i(x \cdot \lambda\varphi) \\ &= \lambda i(x \cdot \varphi) \\ &= \lambda((J_\varphi x) \cdot \varphi) \\ &= (J_\varphi x) \cdot \lambda\varphi. \end{aligned}$$

∎

What this shows is that the complex structure J_φ induced on V by a spinor φ depends only on the orbit of φ under the action of the multiplicative group $\mathbb{C}^*$, i.e. the set of all complex structures compatible with the metric and orientation of V is in $1-1$ correspondence with $\mathbb{P}(S_+)$, the space of projectivised self dual spinors (see [7]).

1.1.2 Spin structures on four-manifolds

Ideally, given a four-manifold X, we would like to put a spin structure on each tangent space in such a way as to vary smoothly across the manifold. This is always possible locally, i.e. within a coordinate patch on the manifold. However, this is not always possible globally. The obstruction to a global spin structure is the second Stiefel-Whitney class

$$w_2 \in H^2(X; \mathbb{Z}_2).$$

If this is zero and X is orientable, then a local spin structure can be extended over the whole manifold and the manifold is said to be spin. The number of global spin structures on a spin manifold is in $1-1$ correspondence with elements of $H^1(X; \mathbb{Z}_2)$, so by the Hurewicz theorem, a simply-connected spin four manifold admits a unique spin structure.

We now present an equivalent way of saying this (see [21]). On a four dimensional Riemannian spin manifold X, we have the frame bundle which is a principal SO(4)-bundle over X . We shall denote this by P_{SO}. A spin structure is a lift from P_{SO} to a principal Spin(4)-bundle, P_{Spin} over X, the obstruction to this lift being the Stiefel-Whitney class w_2 (this is best seen by using Čech cohomology [27]). Given a representation $\rho\colon \mathrm{Spin}(4) \longrightarrow \mathrm{End}(S_+ \oplus S_-)$ we may form the spin bundles

$$W^{\pm} = P_{\mathrm{Spin}} \times_\rho S_{\pm}.$$

(If the manifold is not spin, then W^+ and W^- are 'virtual bundles', but the bundles $W^+ \otimes W^-$ and $\mathbb{P}(W^{\mp})$ always exist.)

We also get an extension of $\gamma\colon \mathrm{T}X \longrightarrow \mathrm{Hom}(W^+, W^-)$. Because we are on a Riemannian manifold, we may identify $\mathrm{T}X$ with T^*X using the Riesz isomorphism

$$\begin{array}{ccc} \mathrm{T}X & \longrightarrow & \mathrm{T}^*X \\ v & \mapsto & \langle v, \cdot \rangle . \end{array}$$

So composing this with γ, we have an action of T^*X and so we can regard forms on X acting on the spin bundles.

On X, we may form the Levi-Civita connection $\nabla\colon \Gamma(\mathrm{T}X) \longrightarrow \Gamma(\mathrm{T}^*X \otimes \mathrm{T}X)$ This is the same as choosing an invariant vertical form $A \in \Omega^1(\mathrm{P}_{\mathrm{SO}}(X); \mathfrak{so}(4))$. Since we make a lifting $\mathrm{SO}(4) \longrightarrow \mathrm{Spin}(4)$, have an isomorphism $\mathfrak{so}(4) \longrightarrow \mathfrak{spin}(4)$,and hence can regard $A \in \Omega^1(\mathrm{P}_{\mathrm{Spin}}(X); \mathrm{End}(W^+ \oplus W^-))$. In this way, the Levi Civita connection induces a connection ∇ on W^+ and W^-.

Now,

$$\nabla\colon \Gamma(W^{\pm}) \longrightarrow \Gamma(\mathrm{T}^*X \otimes W^{\pm})$$

and we may contract this to an operator

$$\mathrm{D}\colon \Gamma(W^+ \oplus W^-) \longrightarrow \Gamma(W^- \oplus W^+)$$

where a 1-form η acts via

$$\begin{array}{lll} \eta & \mapsto & \gamma(\eta) \quad \text{on } W^+ \\ \eta & \mapsto & -\gamma(\eta)^* \quad \text{on } W^-. \end{array}$$

Given a local orthonormal frame $\{e_i\}_{i=1}^4$, D can be written

$$\mathrm{D}(s_1, s_2) = (\gamma(e_i)\nabla_{e_i}s_1, -\gamma(e_i)^*\nabla_{e_i}s_2).$$

This is called the Dirac operator and this can be split into two operators

$$\not{D}\colon \Gamma(W^+) \longrightarrow \Gamma(W^-)$$

and its adjoint. In local coordinates and in a local orthonormal frame

$$\not{D}s = \gamma(e_i)\nabla_{e_i}s, \qquad \not{D}^*s = -\gamma(e_i)^*\nabla_{e_i}s.$$

The total Dirac operator can then be written

$$\mathrm{D} = \begin{pmatrix} 0 & \not{D}^* \\ \not{D} & 0 \end{pmatrix}.$$

1.1.3 Coupling Vector Bundles with Spin Bundles

Let E be an $\mathrm{U}(n)$-bundle over S^4 (i.e. a rank n complex vector bundle with hermitian inner product). This forms a trivial vector bundle $\underline{E}$ over $\mathbb{R}^4$ by the identification of $\mathbb{R}^4$ with $\mathrm{S}^4\backslash\infty$. Let ∇_A be an anti-self dual (ASD) connection on E with covariant derivative ∇_A. This pulls back to an ASD $\mathrm{U}(n)$-connection on $\underline{E}$ over $\mathbb{R}^4$ with finite L^2 energy ($\|F_A\| < \infty$) since the ASD condition and the finite energy condition are both conformally invariant.

Conversely, given a trivial $\mathrm{U}(n)$-bundle $\underline{E}$ over $\mathbb{R}^4$ and a finite energy, ASD connection on it, we can recover an $\mathrm{U}(n)$-bundle E over S^4 by Uhlenbeck's Removable Singularities theorem (see [13]). Let $\{e_i\}_{i=1}^4$ be an orthonormal basis of $\mathbb{R}^4$ and endow $\mathbb{R}^4$ with the complex structure given by

$$\begin{array}{ll} e_1 \mapsto e_2, & e_2 \mapsto -e_1, \\ e_3 \mapsto e_4, & e_4 \mapsto -e_3. \end{array}$$

Associated to $\mathbb{R}^4$ are the spin spaces S^+ and S^- and these decompose with respect to the complex structure into orthogonal subspaces. Let $\{\sigma_i^\pm\}_{i=1}^2$ be a unitary basis for $S^\pm$ with respect to this decomposition. Let

$$\gamma\colon \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathrm{Hom}(S^+, S^-)$$

be the generating homomorphism of the Clifford algebra of $\mathbb{R}^4$. That is, we have an embedding

$$\begin{array}{c} \mathbb{R}^4 \longrightarrow \mathrm{End}(S^+ \oplus S^-) \\ v \mapsto \begin{pmatrix} 0 & -\gamma(v)^* \\ \gamma(v) & 0 \end{pmatrix}. \end{array}$$

Set $q_i = \gamma(e_i)$ and denote Clifford multiplication by $v \cdot \phi$ on the total spin space $S^+ \oplus S^-$. Notice that

$$Q_2^{\pm} = \mathbb{i} = \tfrac{1}{2}(e_1 \cdot e_2 \pm e_3 \cdot e_4),$$
$$Q_3^{\pm} = \mathbb{j} = \tfrac{1}{2}(e_1 \cdot e_3 \pm e_4 \cdot e_2),$$
$$Q_4^{\pm} = \mathbb{k} = \tfrac{1}{2}(e_1 \cdot e_4 \pm e_2 \cdot e_3)$$

as elements of $\mathrm{Cl}_{\mathrm{even}}(\mathbb{R}^4)$ all act trivially on $S^{\mp}$.Also set $Q_1^{\pm} = \mathbb{1}\colon S^{\pm} \longrightarrow S^{\pm}$ This gives an action of the quaternions as endomorphisms of $S^{\pm}$. Where necessary, we will be using the spin structure which,in the given bases of $S^{\pm}$, is given by

$$\gamma(x) = \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix},$$

though most of the results will obviously not depend on which spin structure we choose, as on $\mathbb{R}^4$ and on S^4 all spin structures are equivalent. Now, let

$$f\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$$
$$f(x) = \tfrac{1}{1+|x|^2}.$$

We obtain the well known conformal diffeomorphism of Riemannian manifolds $(\mathbb{R}^4, g) \longrightarrow (\mathrm{S}^4 \backslash \infty, can)$, where $g_x\colon \mathrm{T}_x\mathbb{R}^4 \otimes \mathrm{T}_x\mathbb{R}^4 \longrightarrow \mathbb{R}$ is the metric

$$g_x(v, w) = 4f(x)^2 \langle v, w \rangle .$$

Consequently the spin representation for $\mathrm{S}^4 \backslash \infty$ is given by

$$(\gamma_{\mathrm{S}})_p\colon \mathrm{T}_p(\mathrm{S}^4 \backslash \infty) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(S^+, S^-)$$
$$(\gamma_{\mathrm{S}})_p = 2f(p')\gamma$$

where $p' \in \mathbb{R}^4$ is the point corresponding to $p \in \mathrm{S}^4 \backslash \infty$. Now, let W^+ and W^- be the spinor bundles for S^4. Restricted to $\mathrm{S}^4 \backslash \infty \cong \mathbb{R}^4$, these are the trivial bundles $\underline{S^+}$ and $\underline{S^-}$ respectively.

Unless stated, $\otimes = \otimes_{\mathbb{C}}$.

Definition 1.1.2 *We define* ADHM data *to be a collection of the following*

1. *a* $\mathrm{U}(k)$*-representation space* $\mathcal{H}$,
2. *four self-adjoint endomorphisms* $T_i\colon \mathcal{H} \longrightarrow \mathcal{H}$,
3. *a linear map* $P\colon E_\infty \longrightarrow \mathcal{H} \otimes S^+$;

such that

1. $0 = [T, T]^+ + (PP^*)_0$, *where* $(PP^*)_0$ *is that part of* PP^* *that lies in* $\mathrm{End}(\mathbb{C}^k) \otimes \mathfrak{su}(\underline{S^+})$;

2. *the map*

$$\mathcal{R}\colon \underline{\mathcal{H}\otimes S^- \oplus E} \longrightarrow \underline{\mathcal{H}\otimes S^+}$$

of bundles trivial over $\mathbb{R}^4$, *defined by*

$$\mathcal{R}_x = ((T_i - x_i \mathbb{1}_{\mathbb{C}^k})\otimes q_i^*, P)$$

is surjective.

Each set of ADHM data yields a bundle $\hat{E} = \ker \mathcal{R}$ over $\mathbb{R}^4$ isomorphic to $\underline{E}$ via $v\colon \underline{E}\longrightarrow \hat{E}$, together with an ASD connection given by

$$\Omega^0(\mathbb{R}^4;\underline{E}) \xrightarrow{v} \Omega^0(\mathbb{R}^4;\underline{\mathcal{H}\otimes S^- \oplus E}) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{R}^4;\underline{\mathcal{H}\otimes S^- \oplus E}) \xrightarrow{v^*} \Omega^1(\mathbb{R}^4;\underline{E}).$$

Denote this connection by $A(T,P)$. We now calculate the second Chern class of $\hat{E}$.

Lemma 1.1.3 *The second Chern class of* $\hat{E}$ *over* S^4 *is* k.

Proof

(After [3, 13]) Identify $\mathbb{R}^4$ in the usual way with $\mathbb{H}$. We have been using the bundle map

$$\mathcal{R}\colon \underline{\mathbb{C}^k\otimes S^- \oplus \mathrm{E}_\infty} \longrightarrow \underline{\mathbb{C}^k\otimes S^+}$$

of bundles trivial over $\mathbb{R}^4$ defined by

$$\mathcal{R}_x = ((T_i - x_i \mathbb{1}_{\mathbb{C}^k})\otimes q_i^*, P).$$

We may regard S^+ and S^- as copies of $\mathbb{H}$ and that the spin structure γ^* acts as left multiplication. We may then define $\mathcal{R}^{\mathbb{P}}\colon \underline{\mathbb{C}^k\otimes \mathbb{H}\oplus \mathrm{E}_\infty}\longrightarrow\underline{\mathbb{C}^k\otimes\mathbb{H}}$, a map of bundles trivial over $\mathbb{R}^4\times\mathbb{R}^4 = \mathbb{H}^2$ by

$$(\mathcal{R}^{\mathbb{P}})_{(x,y)} = (yT - x\mathbb{1}_{\mathbb{C}^k}, yP)$$

where the Q_j^+ are the quaternions acting as described in Section 1.1. Let $h\in\mathbb{H}$,then

$$(\mathcal{R}^{\mathbb{P}})_{(hx,hy)} = h(\mathcal{R}^{\mathbb{P}})_{(x,y)}$$

and so the kernel of $(\mathcal{R}^{\mathbb{P}})_{(hx,hy)}$ can be identified with the kernel of $(\mathcal{R}^{\mathbb{P}})_{(x,y)}$ for any $(x,y)\in\mathbb{H}^2$. Therefore we can divide out by the equivalence relation

$$(x,y)\sim(x',y') \iff \exists h\in\mathbb{H} \quad \text{such that}$$

$$x' = hx \quad \text{and} \quad y' = hy$$

to get $\mathcal{R}^{\mathbb{P}}\colon \mathbb{U}\longrightarrow\mathbb{V}$, a map of bundles over $\mathbb{HP}^1 = \mathrm{S}^4$.

Now the trivial bundle $\underline{\mathbb{C}^k\otimes S^-\oplus E}\longrightarrow \mathbb{HP}^1$ decomposes as

$$\underline{\mathbb{C}^k\otimes S^-\oplus E} \cong \ker\mathcal{R}^{\mathbb{P}} \oplus \mathrm{im}\,(\mathcal{R}^{\mathbb{P}}).$$

The kernel of $\mathcal{R}^{\mathbb{P}}$ is precisely the extended bundle generated by the kernel of $\mathcal{R}$ hence isomorphic to $\hat{E}$ over S^4. Now $\mathcal{R}^{\mathbb{P}}$ is surjective, and the image of $(\mathcal{R}^{\mathbb{P}})_{(x,y)}$ is isomorphic to k copies of the line $\mathbb{H}(x,y)$. Hence

$$\mathrm{im}\,(\mathcal{R}^{\mathbb{P}}) \cong \bigoplus\nolimits_{l=1}^{k} K,$$

where K is the tautological line bundle over $\mathbb{HP}^1$ whose fibre over the line l is the line l. we have an isomorphism of bundles over S^4

$$\underline{\mathbb{C}^k \otimes S^- \oplus \mathrm{E}_\infty} \cong \hat{E} \oplus \bigoplus\nolimits_{l=1}^{k} K.$$

Hence equating the total Chern class we have

$$1 = c(\hat{E})c(K)^k = (1 + c_2(\hat{E}))(1 + kc_2(K))$$

and

$$(1 + c_2(\hat{E})) = (1 - kc_2(K)) \implies c_2(\hat{E}) = -kc_2(K).$$

Using the standard conventions $c_2(K) = -1$, so the result follows. ∎

This shows the purpose of the parameter k in the data. We now state the famous theorem about the ADHM construction.

Theorem 1.1.4 *Any ASD connection on $\underline{E}$ over $\mathbb{R}^4$ with finite energy is $A(T,P)$ for some ADHM data. Further, the bundle E over S^4 will have*

$$\int_{S^4} c_2(E) = k.$$

Our goal is to prove this theorem.

1.2 Conformal Invariance

1.2.1 The Dirac Operator

The Dirac operator, $\not{D}$, on the spinor bundles over $\mathbb{R}^4$ can be coupled with A to give

$$\not{D}_A : \Omega^0(\mathbb{R}^4; \underline{E \otimes S^+}) \longrightarrow \Omega^0(\mathbb{R}^4; \underline{E \otimes S^-})$$

lefined on simple elements by

$$\not{D}_A(h \otimes \sigma) = (\nabla_A)_i h \otimes q_i\sigma + h \otimes \not{D}\sigma.$$

his has formal adjoint

$$\not{D}_A^* : \Omega^0(\mathbb{R}^4; \underline{E \otimes S^-}) \longrightarrow \Omega^0(\mathbb{R}^4; \underline{E \otimes S^+}).$$

By the Weitzenböck-Lichnerowitz formula, we have

$$\not{D}_A^*\not{D}_A = \nabla_A^*\nabla_A - F_A^+ + \frac{s}{4} \tag{1.1}$$

where F_A^+ is the self dual part of the curvature of ∇_A and s is the scalar curvature of $\mathbb{R}^4$. We have $F_A^+ = 0$ since A is ASD and $s = 0$ since $\mathbb{R}^4$ is flat. Hence we have

$$\not{D}_A^*\not{D}_A = \nabla_A^*\nabla_A. \tag{1.2}$$

We consider solutions of

$$\not{D}_A^*\psi = 0, \quad \psi \in L^2\Omega^0(\mathbb{R}^4; \underline{E \otimes S^+}). \tag{1.3}$$

The Dirac operator, $\not{D}_{A\mathrm{S}}$, on S^4 is related to $\not{D}_A$ by

$$f^{-\frac{1}{2}}\,\not{D}_A^* f^{\frac{3}{2}} = \not{D}_{A\mathrm{S}}^*. \tag{1.4}$$

So an element in the kernel of $\not{D}_A^*$ corresponds to an element of the kernel of $\not{D}_{A\mathrm{S}}^*$ and vice versa. Also, sections of E over S^4 correspond to L^2 sections of $\underline{E}$ over $\mathbb{R}^4$. Now $\not{D}_{A\mathrm{S}}^*$ is an elliptic operator on a compact manifold, so has finite dimensional kernel. Pulling this back to $\mathbb{R}^4$, $\not{D}_A^*$ has a finite dimensional kernel.

By the Atiyah-Singer index theorem

$$\dim\ker\not{D}_A^* = \operatorname{ind}\not{D}_A^* = \#c_2(E) = k,$$

since $\not{D}_A$ is injective. Hence we can find an L^2-unitary basis, $\{\psi_i\}_{i=1}^k$ of

$$\mathcal{H} = \ker\not{D}_A^*\Big|_{L^2\Omega^0(\mathbb{R}^4;\underline{E\otimes S^-})}.$$

On S^4, set $\mathcal{L}_A$ to be the conformal connection Laplacian (CCL). This is given by

$$\mathcal{L}_A = \nabla_A^{(*)}\nabla_A + 2,$$

where $(*)$ denotes the adjoint with respect to the spherical metric, and conformally transforms to $f\nabla_A^*\nabla_A f^{-1}$ on $\mathbb{R}^4$. The operator $\mathcal{L}_A$ is elliptic, formally self-adjoint and strictly positive, hence is invertible with integral kernel k_A which satisfies

$$k_A(x,y) \sim d(x,y)^{-2}$$

when x is close to y. From this fact we get two useful ideas:

1. We obtain an integral operator G_A with kernel derived from k_A (we just put finite values in k_A) which has the property that over $\mathbb{R}^4$

$$\nabla_A^*\nabla_A G_A s = s$$

for any smooth compactly-supported section s of $\underline{E}$.

2. Given any $v \in E$, there is a unique smooth covariant constant section s of $\underline{E}$ over $\mathbb{R}^4$ such that $s_v(x) = v + O(|x|^{-2})$ and $\|\frac{\partial s_v}{\partial x^i}(x)\| = O(|x|^{-1})$. We obtain this by pulling back the solution on S^4 whose value at infinity is v. However, these sections are not in $L^2\Omega^0(\mathbb{R}^4; \underline{E})$, so we multiply them by the scaling factor $f^{\frac{3}{2}}$ to obtain for any $v \in E$, a unique smooth section s of $\underline{E}$ over $\mathbb{R}^4$ such that $s_v(x) = O(|x|^{-3})$ and $\|\frac{\partial s_v}{\partial x^i}(x)\| = O(|x|^{-4})$, such that

$$\lim_{|x| \longrightarrow \infty} f^{-\frac{3}{2}} s_v(x) = v.$$

1.2.2 The space $\mathcal{H}$

Let $\mathcal{H}$ be the space of solutions of (1.3) and $\{\psi_i\}_{i=1}^k$ be a unitary basis of $\mathcal{H}$. Define

$$\Psi : \mathbb{C}^k \longrightarrow \Omega^0(\mathbb{R}^4; \underline{E \otimes S^-})$$

as the obvious embedding

$$\Psi z = z^i \psi_i. \tag{1.5}$$

The adjoint to this is an integral operator defined by

$$\langle \Psi^* \phi, z \rangle_{\mathbb{C}^k} = \langle \phi, \Psi z \rangle_{L^2} = \int_{\mathbb{R}^4} \langle \phi, \Psi z \rangle_{E \otimes W^-} \, dvol \tag{1.6}$$

for all $z \in \mathbb{C}^k$. So we have the identities

$$\Psi^* \Psi = \mathbb{1}_{\mathbb{C}^k}, \tag{1.7}$$

$$\not{D}_A^* \Psi = 0, \tag{1.8}$$

$$(\Psi \Psi^*)^2 = \Psi \Psi^*. \tag{1.9}$$

Equation (1.9) tells us that $\Psi\Psi^*$ is unitary projection onto $\operatorname{im}\Psi = \mathcal{H}$. But this unitary projection is also obviously given by the integral operator

$$\mathbb{1}_{\Omega^0(\mathbb{R}^4; E \otimes W^-)} - \not{D}_A (\not{D}_A^* \not{D}_A)^{-1} \not{D}_A^* = \mathbb{1}_{\Omega^0(\mathbb{R}^4; E \otimes W^-)} - \not{D}_A G_A \not{D}_A^*, \tag{1.10}$$

since by (1.2)

$$(\not{D}_A^* \not{D}_A)^{-1} = (\nabla_A^* \nabla_A)^{-1} = G_A.$$

Now, we want to consider the behaviour of Ψ near ∞.

Any solution of (1.3) on $\mathbb{R}^4$ satisfies

$$|\psi(x)| = O(|x|^{-3}).$$

This is because ψ corresponds to a solution of

$$\not{D}_A^* s = 0$$

on S^4 by conformal invariance which has bounded pointwise norm, so

$$\begin{aligned} |\psi(x)| &= |f(x)^{\frac{3}{2}} s(x)| \leq \frac{const}{(1+|x|^2)^{\frac{3}{2}}} \\ &\leq \frac{const}{|x|^3}. \end{aligned}$$

Following [13], we will write $|s(x)| = \mathrm{O}'(|x|^{-l})$ if $|s(x)| = \mathrm{O}(|x|^{-l})$ and $|\mathrm{d}s(x)| = \mathrm{O}(|x|^{-l-1})$.

Given any $\phi \in \mathcal{H}$, we can transform it back to the sphere and evaluate it at infinity. Consequently, we have a natural evaluation map $\mathrm{ev}\colon \mathcal{H} \longrightarrow (E \otimes W^-)_\infty$.

1.3 Spinors at infinity

1.3.1 Spinors on S^4

Now, regarding S^4 as $\mathbb{HP}^1$, i.e.

$$S^4 = \frac{\mathbb{H} \sqcup \mathbb{H}}{\sim}$$

where for any $x, y \in \mathbb{H}$

$$x \sim y \iff y \neq 0 \quad \text{and} \quad x = y^{-1}.$$

The identification is a natural orientation reversal, and the transition map is

$$x \mapsto \frac{x}{||x||^2}$$

Using the orientation reversing map between the tangent spaces to the sphere at 0 and ∞, we obtain an identification $\varepsilon\colon (W^+ \oplus W^-)_\infty \longrightarrow \underline{S^- \oplus S^+}_0$, since the orientation reversal reverses the action of Spin(4). That is, the fibre at infinity of $W^\pm$ is the space $S^\mp$. In particular, for any $\rho \in E \otimes W^-$ at ∞ and any $\sigma \in \mathrm{CL}_{\mathrm{even}}(\mathbb{R}^4)$, $\varepsilon(\sigma + \star\sigma)\rho$ is identified with $(\sigma - \star\sigma)\varepsilon\rho$ in $E \otimes W^+$ at 0.

Proposition 1.3.1 *There is a linear map $\tilde{\alpha}\colon \mathbb{C}^k \longrightarrow E \otimes W^-{}_\infty$ such that near ∞ (i.e. outside some sufficiently large compact set containing 0), Ψ has the form*

$$\Psi = \frac{x\varepsilon\tilde{\alpha}}{\pi r^4} + \mathrm{O}'(r^{-4}).$$

Proof
For any $h \in S^+$, the singular section $s_h \in \Omega^0(S^4; W^-)$ given by

$$s_h = \frac{x}{r^4} h$$

satisfies $\not{D}^* s_h = 0$ and is $\mathrm{O}'(|x|^{-3})$ near infinity. Now the evaluation map ev tells us the behaviour of any $\phi \in \mathcal{H}$ near infinity and specifically tells us the part of ϕ that varies as $|x|^{-3}$. So if $\mathrm{ev}(\psi) = \rho \otimes h$ for $\psi \in \mathcal{H}$ then, near infinity, ψ has the form

$$\psi(x) = \rho \otimes \frac{x}{\pi r^4} h + \mathrm{O}'(|x|^{-4}). \tag{1.11}$$

Hence in a neighbourhood of infinity, Ψ has the form

$$\Psi = \frac{x\varepsilon\tilde{\alpha}}{\pi r^4} + \mathrm{O}'(r^{-4}) \tag{1.12}$$

where $\tilde{\alpha} = \mathrm{ev}\Psi \colon \mathbb{C}^k \longrightarrow (E \otimes W^-)_\infty$. ■

1.3.2 Some Spinor Analysis

Although a change of gauge can affect the asymptotic behaviour of a connection, we will always trivialise the bundle E over S^4 in a neighbourhood of ∞ such that

$$\left\| \frac{\partial^{|l|}}{\partial x^l} A \right\| = \mathrm{O}(|x|^{-3-|l|})$$

for a multi-index l of sufficient order.In particular, the curvature has norm $\mathrm{O}(|x|^{-4})$. Also, we can regard $\rho \in \mathrm{E}_\infty \otimes S^+$ as a section of $\underline{E \otimes S^+}$ over $\mathbb{R}^4$, via $x \mapsto (x, \rho)$, in which case, in the above local trivialisation

$$\not{D}_A \rho = \mathrm{O}'(|x|^{-3}). \tag{1.13}$$

This allows us to examine the asymptotic behaviour of the Green's operator on harmonic spinors.

Proposition 1.3.2 *Outside a compact subset of* $\mathbb{R}^4$ *containing* 0,

$$G_A \Psi = \frac{1}{4} r^2 \Psi + \mathrm{O}'(r^{-2}). \tag{1.14}$$

Proof
We find $\not{D}_A \not{D}_A^*$ has the form

$$\not{D}_A \not{D}_A^* = \nabla_A^* \nabla_A + \text{curvature terms} \tag{1.15}$$

by another Weitzenböck argument. Also, for $e \in \Omega^0(\mathrm{S}^4; E \otimes W^-)$, we have

$$\not{D}_A \not{D}_A^* \frac{x}{r^2} e = \frac{4x}{r^4} e - \frac{2}{r^2} \not{D}_A e + \frac{x}{r^2} \not{D}_A^* \not{D}_A e + 2 \frac{x_\mu}{r^2} q_\nu (F_A)_{\mu\nu} e. \tag{1.16}$$

So using (1.11) and (1.13) for $\phi \in \mathcal{H}$ with $\mathrm{ev}(\phi) = \rho$, near infinity

$$\begin{aligned}\phi(x) &= \frac{x}{\pi r^4}\rho + \mathrm{O}'(|x|^{-4}) \qquad \text{by (1.3.1)} \\ &= \not{D}_A \not{D}_A^* \frac{x}{4\pi r^2}\rho + \frac{2}{\pi r^2}\not{D}_A \rho \\ &\quad - \frac{x}{\pi r^2} \not{D}_A^* \not{D}_A \rho - 2\frac{x_\mu}{\pi r^2} q_\nu (F_A)_{\mu\nu}\rho + \mathrm{O}'(|x|^{-4})\end{aligned}$$

using (1.16)

$$= \nabla_A^* \nabla_A \frac{x}{4\pi r^2}\rho + \mathrm{O}'(|x|^{-4}).$$

Now apply the Green operator to both sides to get

$$G_A\phi(x) = \frac{x}{4r^2}\rho + \mathrm{O}'(|x|^{-2}) = \frac{r^2}{4}\phi(x) + \mathrm{O}'(|x|^{-2}). \tag{1.17}$$

Thus we have the identity

$$G_A\Psi = \frac{1}{4}r^2\Psi + \mathrm{O}'(r^{-2})$$

as required. ∎

1.4 Setting up the data

1.4.1 The map T

Let $T_i\colon \mathbb{C}^k \longrightarrow \mathbb{C}^k$, $i = 1 \ldots 4$ be the unitary transformations given by

$$T_i = -\Psi^* x_i \Psi, \tag{1.18}$$

understanding that we first project $x_i\Psi$ to the L^2 sections. We examine the Lie brackets of these transformations

$$[T_i, T_j] = \Psi^* x_{[i} \Psi \Psi^* x_{j]} \Psi \tag{1.19}$$

where

$$a_{[i}b_{j]} = a_i b_j - a_j b_i. \tag{1.20}$$

Lemma 1.4.1

$$[T_i, T_j]z = -\Psi^* q_{[i} q^*{}_{j]} G_A \Psi z - \frac{1}{8}\tilde{\alpha}^* \varepsilon^* (q^*{}_{[i} q_{j]}) \varepsilon \tilde{\alpha} z. \tag{1.21}$$

Proof
We have

$$\not{D}_A x_i s = q_i s + x_i \not{D}_A s \quad \text{and} \quad \not{D}_A^* x_i s = -q_i^* s + x_i \not{D}_A^* s. \tag{1.22}$$

Using (1.10) and setting $\{\zeta_\mu\}_{\mu=1}^k$ to be the canonical unitary basis of $\mathbb{C}^k$, we find for each $z \in \mathbb{C}^k$

$$\begin{aligned}
[T_i, T_j]z &= -\Psi^* x_{[i} \not{D}_A G_A \not{D}_A^* x_{j]} \Psi = \Psi^* x_{[i} \not{D}_A G_A q_{j]}^* \Psi \\
&\qquad \text{using (1.22) and (1.8)} \\
&= \int_{\mathbb{R}^4} \left\langle x_{[i} \not{D}_A G_A q_{j]}^* \Psi z, \Psi \zeta_\mu \right\rangle_{E \otimes W^-} dvol \zeta_\mu \\
&\qquad \text{using the formula for } \Psi^* \\
&= -\int_{\mathbb{R}^4} \left\langle \not{D}_A G_A q_{[j}^* \Psi z, x_{i]} \Psi \zeta_\mu \right\rangle_{E \otimes W^-} dvol \zeta_\mu .
\end{aligned}$$

Now, we have

$$\frac{\partial}{\partial x^\nu} \langle q_\nu s, t \rangle_{E \otimes W^-} = \langle \not{D}_A s, t \rangle_{E \otimes W^-} - \langle s, \not{D}_A^* t \rangle_{E \otimes W^+} . \tag{1.23}$$

To see this, just expand the lhs and use the fact that A is compatible with the metric. Hence by (1.23) we have

$$\begin{aligned}
[T_i, T_j]z &= -\int_{\mathbb{R}^4} \left\langle q_{[j}^* G_A \Psi z, \not{D}_A^* x_{i]} \Psi \zeta_\mu \right\rangle_{E \otimes W^+} dvol \cdot \zeta_\mu \\
&\quad - \int_{\mathbb{R}^4} \frac{\partial}{\partial x^\nu} \langle q^*{}_\nu q_{[j} G_A \Psi z, x_{i]} \Psi \zeta_\mu \rangle_{E \otimes W^-} dvol \cdot \zeta_\mu \\
&= \int_{\mathbb{R}^4} \left\langle q_{[j}^* G_A \Psi z, q_{i]}^* \Psi \zeta_\mu \right\rangle_{E \otimes W^+} dvol \cdot \zeta_\mu \\
&\quad - \lim_{R \longrightarrow \infty} \int_{S_R(0)} \left\langle \frac{x}{R} q_{[j}^* G_A \Psi z, x_{i]} \Psi \zeta_\mu \right\rangle_{E \otimes W^-} dA_R \cdot \zeta_\mu \\
&\quad \text{by Stokes' theorem and (1.22).}
\end{aligned}$$

Thus

$$\begin{aligned}
[T_i, T_j]z &= -\Psi^* q_{[i} q_{j]}^* G_A \Psi z + \lim_{R \longrightarrow \infty} \int_{S_R(0)} \left\langle x_{[i} \frac{x}{R} q_{j]}^* G_A \Psi z, \Psi \zeta_\mu \right\rangle_{E \otimes W^-} dA_R \cdot \zeta_\mu . \\
&= -\Psi^* q_{[i} q_{j]}^* G_A \Psi z + \lim_{R \longrightarrow \infty} \int_{S_R(0)} \left\langle x_{[i} \frac{x}{R} q_{j]}^* G_A \Psi z, \frac{x \varepsilon \alpha}{\pi R^4} \zeta_\mu \right\rangle_{E \otimes W^-} dA_R \cdot \zeta_\mu \\
&\quad + \lim_{R \longrightarrow \infty} O(R^{-1}),
\end{aligned}$$

and by Prop (1.3.1)

$$[T_i, T_j]z = -\Psi^* q_{[i} q_{j]}^* G_A \Psi z + \lim_{R \longrightarrow \infty} \int_{S_R(0)} \left\langle x_{[i} q_{j]}^* G_A \Psi z, \frac{\varepsilon \tilde{\alpha}}{\pi R^2} \zeta_\mu \right\rangle_{E \otimes W^-} dA_R \cdot \zeta_\mu .$$

By (1.14) and Prop (1.3.1)

$$\begin{aligned}
[T_i, T_j]z &= -\Psi^* q_{[i}q_{j]}^* G_A \Psi z \\
&\quad + \lim_{R\longrightarrow\infty} \int_{\mathrm{S_R(0)}} \left\langle x_{[i}q_{j]}^* \frac{x\varepsilon\tilde{\alpha}}{4\pi R^2} z, \frac{\varepsilon\alpha}{\pi R^3}\zeta_\mu \right\rangle_{E\otimes W^-} \mathrm{d}A_R \cdot \zeta_\mu \\
&= -\Psi^* q_{[i}q_{j]}^* G_A \Psi z \\
&\quad + \frac{1}{4\pi^2} \lim_{R\longrightarrow\infty} \int_{\mathrm{S_R(0)}} \left\langle \frac{1}{R^5} x_{[i}q_{j]}^* x\varepsilon\tilde{\alpha} z, \varepsilon\tilde{\alpha}\zeta_\mu \right\rangle_{E\otimes W^-} \mathrm{d}A_R \cdot \zeta_\mu .
\end{aligned}$$

Then,

$$\begin{aligned}
[T_i, T_j]\, z &= -\Psi^* q_{[i}q_{j]}^* G_A \Psi z \\
&\quad + \frac{1}{4\pi^2} \lim_{R\longrightarrow\infty} \int_{\mathrm{S^3}} \left\langle \frac{1}{R^5} R\theta_{[i}q_{j]}^* R\theta_l q_l \varepsilon\tilde{\alpha} z, \varepsilon\tilde{\alpha}\zeta_\mu \right\rangle_{E\otimes W^+} R^3 \mathrm{d}\theta \cdot \zeta_\mu \\
&= -\Psi^* q_{[i}q_{j]}^* G_A \Psi z + \frac{1}{4\pi^2} \int_{\mathrm{S^3}} \left\langle \theta_l \theta_{[i} q_{j]}^* q_l \varepsilon\tilde{\alpha} z, \varepsilon\tilde{\alpha}\zeta_\mu \right\rangle_{E\otimes W^+} R^3 \mathrm{d}\theta \cdot \zeta_\mu \\
&= -\Psi^* q_{[i}q_{j]}^* G_A \Psi z + \frac{1}{4\pi^2} \int_{\mathrm{S^3}} \theta_l \theta_{[i} \mathrm{d}\theta \left\langle q_{j]}^* q_l \varepsilon\tilde{\alpha} z, \varepsilon\tilde{\alpha}\zeta_\mu \right\rangle_{E\otimes W^+} R^3 \mathrm{d}\theta \cdot \zeta_\mu \\
&= -\Psi^* q_{[i}q_{j]}^* G_A \Psi z - \frac{1}{4\pi^2} \left(\int_{\mathrm{S^3}} \theta_l \theta_{[i} \mathrm{d}\theta \right) \tilde{\alpha}^* \varepsilon^* (q_l^* q_{j]}) \varepsilon\tilde{\alpha} z \\
&\quad - \frac{1}{2\pi^2} \left(\int_{\mathrm{S^3}} \theta_l \theta_{[i} \delta_{lj]} \mathrm{d}\theta \right) \tilde{\alpha}^* \varepsilon^* \varepsilon\tilde{\alpha} z.
\end{aligned}$$

Now

$$\int_{\mathrm{S^3}} \theta_l \theta_m \mathrm{d}A = \frac{\pi^2}{2} \delta_{lm},$$

so we finally obtain

$$[T_i, T_j]z = -\Psi^* q_{[i}q_{j]}^* G_A \Psi z - \frac{1}{8} \tilde{\alpha}^* \varepsilon^* (q_{[i}^* q_{j]}) \varepsilon\tilde{\alpha} z, \tag{1.24}$$

or in more Clifford algebraic form

$$\begin{aligned}
[T_i, T_j]z &= \Psi^* e_{[i} \cdot e_{j]} G_A \Psi z + \tfrac{1}{8} \tilde{\alpha}^* \varepsilon^* (e_{[i} \cdot e_{j]}) \varepsilon\tilde{\alpha} z \\
&= 2\Psi^* e_i \cdot e_j \cdot G_A \Psi z + \tfrac{1}{4} \tilde{\alpha}^* \varepsilon^* e_i \cdot e_j \cdot \varepsilon\tilde{\alpha} z.
\end{aligned} \tag{1.25}$$

■

1.4.2 The map P and the ADHM Equations

Set $\alpha = \varepsilon\tilde{\alpha}\colon \mathbb{C}^k \longrightarrow \mathrm{E}_\infty \otimes S^+$. Since $S^+ \cong \mathbb{C} \oplus \mathbb{C}$, α has the form

$$\alpha z = \alpha_1 z \otimes \sigma_1^+ + \alpha_2 z \otimes \sigma_2^+ \tag{1.26}$$

where $\alpha_1, \alpha_2 \colon \mathbb{C}^k \longrightarrow \mathrm{E}_\infty$ are $\mathbb{C}$-linear maps.

Now calculating each of the $[T_i, T_j]$s explicitly using the representation stated in Section 1.1.1, we find

Proposition 1.4.2

$$[T_1,T_2] + [T_3,T_4] = -\tfrac{i}{2}\left(\alpha_1{}^*\alpha_1 - \alpha_2{}^*\alpha_2\right) \tag{1.27}$$
$$[T_1,T_3] + [T_4,T_2] = -\tfrac{1}{2}\left(\alpha_2{}^*\alpha_1 - \alpha_1{}^*\alpha_2\right) \tag{1.28}$$
$$[T_1,T_4] + [T_2,T_3] = -\tfrac{i}{2}\left(\alpha_2{}^*\alpha_1 + \alpha_1{}^*\alpha_2\right). \tag{1.29}$$

Define $P \colon \mathrm{E}_\infty \longrightarrow \mathbb{C}^k \otimes S^+$ to be the map

$$\alpha_1{}^* \otimes \sigma_2^+ - \alpha_2{}^* \otimes \sigma_1^+.$$

This is just the image of α^* under the isomorphism

$$(\mathrm{E}_\infty \otimes S^+)^* \otimes \mathbb{C}^k = \mathrm{E}_\infty{}^* \otimes \left(S^+\right)^* \otimes \mathbb{C}^k \cong \mathrm{E}_\infty{}^* \otimes S^+ \otimes \mathbb{C}^k$$

via the symplectic form on S^+. We have the first data for the ADHM construction according to [13], that is:

ADHM data	Our situation
a $\mathrm{U}(k)$-representation space $\mathcal{H}$	$\mathbb{C}^k \cong \mathcal{H} = \ker \left. \not{D}_A^* \right\|_{L^2\Omega^0(\mathbb{R}^4; E\otimes W^-)}$
self-adjoint endomorphisms $T_i \colon \mathcal{H} \longrightarrow \mathcal{H}$	$\Psi^* x_i \Psi$
a linear map $P \colon \mathrm{E}_\infty \longrightarrow \mathcal{H} \otimes S^+$	$\alpha_1{}^* \otimes \sigma_2^+ - \alpha_2{}^* \otimes \sigma_1^+$

We need to show that we truly have ADHM data.

First the condition that

$$[T,T]^+ + (PP^*)_{\mathrm{End}(\mathbb{C}^k)\otimes\mathfrak{su}(S^+)} = 0.$$

This is a calculation involving our chosen spin representation. We check this componentwise by examining PP^* and its relationship with $\mathbb{i}, \mathbb{j}, \mathbb{k}$ or their spin representatives Q_2^+, Q_3^+, Q_4^+ given earlier. These are given by the matrices

$$Q_2^+ = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \; Q_3^+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; Q_4^+ = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

By taking traces in the spinor parts, we have

$$\begin{aligned}
(PP^*)_{\mathbb{i}} &= \frac{1}{2}\mathrm{tr}_{S^+}\left((Q_1^+)^* PP^*\right) = -\frac{i}{2}(\alpha_1^*\alpha_1 - \alpha_2^*\alpha_2) \\
(PP^*)_{\mathbb{j}} &= \frac{1}{2}\mathrm{tr}_{S^+}\left((Q_2^+)^* PP^*\right) = -\frac{1}{2}(\alpha_2^*\alpha_1 - \alpha_1^*\alpha_2) \\
(PP^*)_{\mathbb{k}} &= \frac{1}{2}\mathrm{tr}_{S^+}\left((Q_3^+)^* PP^s\right) = -\frac{i}{2}(\alpha_2^*\alpha_1 + \alpha_1^*\alpha_2)
\end{aligned}$$

which one may compare with the expressions for $[T_i, T_j]$ given above in equations 1.27-1.29 and readily see that the first ADHM condition is satisfied using the fact that

$$\begin{aligned} [T,T]^+ &= -([T_1,T_2]+[T_3,T_4])\otimes Q_2^+ \\ &\quad -([T_1,T_3]+[T_4,T_2])\otimes Q_3^+ \\ &\quad -([T_1,T_4]+[T_2,T_3])\otimes Q_4^+ . \end{aligned}$$

Hence we have fulfilled the first requirement in Definition (1.1.2).

1.5 Constructing the new connection

1.5.1 The map $\mathcal{R}$ and its properties

Set

$$\mathcal{R}_x = ((T_i - x_i)\otimes q_i^*, P) . \tag{1.30}$$

Proposition 1.5.1 *$\mathcal{R}$ is a surjective bundle map.*

Proof
Suppose that $\mathcal{R}$ is not surjective. We have by (3.4.8) of [13] an orthogonal decomposition of $\mathbb{C}^k$ into $V' \oplus V''$ such that the T_i split as $T_i' \oplus T_i''$ with T_i' commuting with T_j' for each i, j. As Donaldson and Kronheimer explain, such data gives a reducible connection. We are assuming that E is irreducible, and hence $\mathcal{R}$ must be surjective. ■

From this, we see that we have the full ADHM data,

1. $[T,T]^+ + (PP^*)_{\mathrm{End}(\mathbb{C}^k)\otimes\mathfrak{su}(S^+)} = 0$ by Proposition 1.4.2

2. $\mathcal{R}$ is surjective.

Also

$$\mathcal{R}\mathcal{R}^* = [(T_i - x_i)(T_i - x_i) + \frac{1}{2}\alpha^*\alpha]\otimes \mathbb{1}_{\underline{S^+}} \tag{1.31}$$

which is an easy identity to prove using Proposition 1.4.2.

1.5.2 The New Connection

Now we construct the new connection.

Assume for the moment that $\underline{E}$ is isomorphic to $\hat{E}$, we will construct an isomorphism later. Let $v\colon \underline{E} \longrightarrow \underline{\mathbb{C}^k \otimes S^- \oplus E}$ be an embedding of $\underline{E}$ as $\hat{E}$ and set $\mathcal{F} = (\mathcal{R}\mathcal{R}^*)^{-1}$ which is well-defined by surjectivity of $\mathcal{R}$. We have the identities

$$v^* v = \mathbb{1}_{\underline{E}} \tag{1.32}$$

and

$$v v^* = \mathbb{1}_{\underline{\mathbb{C}^k \otimes S^- \oplus E}} - \mathcal{R}^* \mathcal{F} \mathcal{R}, \tag{1.33}$$

which are analogous to equations (1.7) and (1.10). Now define a new connection $\mathcal{D}$ on the bundle $\underline{E}$ by

$$\mathcal{D} = v^* \mathrm{d} v. \tag{1.34}$$

Proposition 1.5.2 *The new connection, $\mathcal{D}$, is anti self-dual.*

Proof
This is yet another calculation.

$$\begin{aligned} [\mathcal{D}_i, \mathcal{D}_j] s &= v^* \frac{\partial}{\partial x_{[i}} v v^* \frac{\partial}{\partial x_{j]}} v = -v^* \frac{\partial}{\partial x_{[i}} \mathcal{R}^* \mathcal{F} \mathcal{R} \frac{\partial}{\partial x_{j]}} v \\ & \quad \text{using (1.33) and} \frac{\partial}{\partial x_{[i}} \frac{\partial}{\partial x_{j]}} = 0. \end{aligned}$$

Now, we have the identity

$$\frac{\partial}{\partial x_l} \mathcal{R}^*(s) = q_l s \oplus 0 = \iota(q_l s) \tag{1.35}$$

where ι is the obvious inclusion. So using this and

$$0 = \frac{\partial}{\partial x_l}(\mathcal{R} v) = \frac{\partial}{\partial x_l}(\mathcal{R}) v + \mathcal{R} \frac{\partial}{\partial x_l}(v), \tag{1.36}$$

we have

$$[\mathcal{D}_i, \mathcal{D}_j] s = v^* \iota q_{[i} q_{j]}^* \mathcal{F} \iota^* v s, \tag{1.37}$$

which is anti self-dual as hoped because $\mathcal{F}$ commutes with the q_i and their adjoints. ■

The next step is to show that this is the covariant derivative associated to A, in order to do this we set up an explicit formula for v.

Proposition 1.5.3 *Let $s \in \Omega^0(\mathbb{R}^4; \underline{E})$ be the pull back of a section of E over S^4. Then*

$$\begin{aligned} & G_A (\nabla_A)_i G_A s + \tfrac{1}{2}(x_i G_A s - G_A x_i s) \\ &= k_A(\infty, x) \mathrm{ev}\left(G_A (\nabla_A)_i G_A s + \tfrac{1}{2}(x_i G_A s - G_A x_i s)\right) \end{aligned} \tag{1.38}$$

Proof
First, we have

$$\Delta_A G_A (\nabla_A)_i G_A s = (\nabla_A)_i G_A s.$$

Then

$$\begin{aligned} \Delta_A (x_i G_A) s = -(\nabla_A)_j (\nabla_A)_j x_i G_A s &= -2(\nabla_A)_i G_A s + x_i s \\ \Longrightarrow \Delta_A (x_i G_A s - G_A x_i s) &= -2(\nabla_A)_i G_A s. \end{aligned}$$

So, we know that

$$G_A(\nabla_A)_i G_A s + \frac{1}{2}(x_i G_A s - G_A x_i s) \in \ker \Delta_A \Big|_{S^4 \setminus \infty}.$$

Let this evaluate to $\rho \in E$ at infinity. Then $k_A(\infty, x)\rho$ is also a solution of

$$\Delta_A \phi = 0, \quad \lim_{x \longrightarrow \infty} \phi(x) = \rho.$$

The result then follows by the uniqueness of solutions. ∎

As an immediate corollary, we have

Proposition 1.5.4

$$G_A q_i^* \not{D}_A G_A q_i^* \Psi = -\frac{1}{4\pi} k_A(\infty, x)\alpha - \frac{1}{2} q_i^* \left[x, G_A\right] q_i^* \Psi. \tag{1.39}$$

Proof
Clearly for all $c \in \mathbb{C}^k$, Ψc satisfies the hypothesis of (1.5.3), so all we need to do is to evaluate

$$G_A q_i^* \not{D}_A G_A q_i^* \Psi + \frac{1}{2} q_i^* \left[x, G_A\right] q_i^* \Psi = G_A \left(q_i^* \not{D}_A G_A q_i^* \Psi - \frac{1}{2} q_i^* x q_i^* \Psi \right) + \frac{1}{2} q_i^* x G_A q_i^* \Psi$$

at infinity. First observe that

$$q_i^* x q_i^* = -2x^* \quad \text{and hence} \quad q_i^* \not{D}_A q_i^* = 2\, \not{D}_A^*. \tag{1.40}$$

Hence,

$$\begin{aligned} q_i^* \not{D}_A G_A q_i^* \Psi - \frac{1}{2} q_i^* x q_i^* \Psi &= \not{D}_A^* G_A \Psi + x^* \Psi \\ &= 2\, \not{D}_A^* \left(\frac{1}{4} r^2 \Psi + \mathrm{O}'(|x|^{-2}) \right) + x^* \Psi \quad \text{by (1.14)} \\ &= -x^* \Psi + \mathrm{O}(|x|^{-3}) + x^* \Psi \\ &= \mathrm{O}(|x|^{-3}). \end{aligned}$$

Hence near infinity,

$$G_A \left(q_i^* \not{D}_A G_A q_i^* \Psi - \frac{1}{2} q_i^* x q_i^* \Psi \right) = \mathrm{O}(|x|^{-1}).$$

Now

$$
\begin{aligned}
\frac{1}{2}q_i^* x G_A q_i^* \Psi &= -x^* G_A \Psi \\
&= -\frac{1}{4}x^* r^2 \Psi + O'(|x|^{-1}) && \text{by (1.14)} \\
&= -\frac{1}{4}x^* \frac{x\alpha}{r^2\pi} + O'(|x|^{-1}) && \text{by (1.3.1)} \\
&= -\frac{\alpha}{4\pi} + O'(|x|^{-1}).
\end{aligned}
$$

Hence

$$
\text{ev}\left(G_A q_i^* \not{D}_A G_A q_i^* \Psi + \frac{1}{2} q_i^* [x, G_A] q_i^* \Psi \right) = -\frac{\alpha}{4\pi},
$$

and the result follows by (1.5.3). ■

This leads to

Proposition 1.5.5

$$
G_A q_i^* \Psi (T_i + y_i \mathbb{1}_{\mathbb{C}^k}) = (y - x)^* G_A \Psi + \frac{1}{4\pi} k_A(\infty, x)\alpha.
$$

Proof

This is just another simple calculation.

$$
\begin{aligned}
G_A q_i^* \Psi (T_i + y_i) &= -G_A \Psi \Psi^* x_i \Psi + y^* G_A \Psi \\
&= -G_A x^* \Psi + G_A q_i^* \not{D}_A G_A \not{D}_A^* x_i \Psi + y^* G_A \Psi && \text{by (1.10)} \\
&= G_A (y - x)^* \Psi - G_A q_i^* \not{D}_A G_A q_i^* \Psi.
\end{aligned}
$$

Hence by (1.39)

$$
\begin{aligned}
G_A q_i^* \Psi (T_i + y_i) &= G_A (y - x)^* \Psi + \frac{1}{4\pi} k_A(\infty, x)\alpha + \frac{1}{2} q_i^* [x, G_A] q_i^* \Psi \\
&= G_A (y - x)^* \Psi - x^* G_A \Psi + G_A x^* \Psi + \frac{1}{4\pi} k_A(\infty, x)\alpha && \text{by (1.40)} \\
&= (y - x)^* G_A \Psi + \frac{1}{4\pi} k_A(\infty, x)\alpha
\end{aligned}
$$

as required. ■

Define $V \colon \Omega^0(\mathbb{R}^4; \underline{E}) \longrightarrow \mathbb{C}^k \otimes S^- \oplus \mathrm{E}_\infty$ by

$$
V^*(c \otimes \phi, e) = 4\pi\kappa(G_A \Psi c \otimes \phi) - k_A(\infty, \cdot)e.
$$

and the associated bundle map $v \colon \underline{E} \longrightarrow \underline{\mathbb{C}^k \otimes S^- \oplus E}$ is given by

$$
v_x^*((c \otimes \phi, e)_x) = V^*(c \otimes \phi, e)(x)
$$

where κ is spinor contraction with the symplectic form.

Lemma 1.5.6

$$v^*\mathcal{R}^* = 0 \tag{1.41}$$

Proof
Let $(c \otimes \phi)_y \in \underline{\mathbb{C}^k \otimes S^+}$, then

$$\mathcal{R}^*(c \otimes \phi)_y = \left((T_i c - y_i c) \otimes q_i \phi, P^*(c \otimes \phi)\right)_y.$$

So

$$v^*\mathcal{R}^*(c \otimes \phi)_y = 4\pi\kappa\big(G_A \Psi (T_i c - y_i c) \otimes q_i \phi\big)(y) - k_A(\infty, y) P^*(c \otimes \phi).$$

Now, the spinor contraction has the following property for any $v \in \mathrm{Cl}(\mathbb{R}^4)$

$$\kappa(a \otimes w \cdot b) = \kappa(w \cdot a \otimes b) \tag{1.42}$$

or, for $a \in S^+$ and $b \in S^-$

$$\kappa(\gamma(w) a \otimes b) = \kappa(a \otimes \gamma(w)^* b) \tag{1.43}$$

for $w \in \mathbb{R}^4$. From this we see that

$$\begin{aligned}
v^*\mathcal{R}^*(c \otimes \phi)_y &= 4\pi\kappa\big(G_A q_i^* \Psi (T_i c + y_i c) \otimes \phi\big)(y) - k_A(y, \infty) P^*(c \otimes \phi) \\
&= 4\pi\kappa\left(\left((y - x)^* G_A \Psi c + \frac{1}{4\pi} k_A(\infty, x) \alpha c\right) \otimes \phi\right)\Bigg|_{x=y} \\
&\quad - k_A(y, \infty) P^*(c \otimes \phi) \\
&\quad \text{by (1.5.5)} \\
&= 0
\end{aligned}$$

since $P^*(c \otimes \phi) = \kappa(\alpha c \otimes \phi)$ and k_A is symmetric. ∎

This formula for v^* is obviously independent of choices of complex structure on $\mathbb{R}^4$ and representation of $\mathrm{Spin}(4)$. We now prove that it is related in a particularly nice way to the kernel of G_A, and from that deduce that it is indeed an isometry of $\underline{E}$ onto $\hat{E}$. We need however a formula for v; this is not so easy to describe independently of complex structure.

In the standard complex structure on $\mathbb{R}^4$, and with respect to the splitting of $S^\pm$ with respect to this complex structure, $V \colon \Omega^0(\mathbb{R}^4; \underline{E}) \longrightarrow \mathbb{C}^k \otimes S^- \oplus \mathrm{E}_\infty$ is given by

$$V(s) = (4\pi {\Psi_1}^* G_A s, 4\pi {\Psi_2}^* G_A s, -G_A s(\infty)) \tag{1.44}$$

Lemma 1.5.7 *Let $f_y(x) = \|x - y\|^2$ then for any compactly supported section $s \in \Omega^0(\mathbb{R}^4; \underline{E})$*

$$(V^* V s)(y) = 4\pi^2 G_A f_y s. \tag{1.45}$$

Proof
Applying Δ_A to the rhs yields

$$-32\pi^2 G_A s(y) - 16\pi^2 \left[x_i, (\nabla_A)_i G_A\right] s(y).$$

We now apply Δ_A to the lhs. First notice that

$$V^* V s(y) = 16\pi^2 (G_A \Psi_i \Psi_i^* G_A) s(y) + k_A(y, \infty) G_A s(\infty),$$

From (1.10), $\Psi_i \Psi_i^* = 2(\mathbb{1} + (\nabla_A)_i G_A (\nabla_A)_i)^1$ and so

$$V^* V s(y) = 32\pi^2 (G_A(\mathbb{1} + (\nabla_A)_i G_A (\nabla_A)_i) G_A) s + k_A(y, \infty) G_A s(\infty)$$

By (1.5.3)

$$\begin{aligned}
\Delta_A V^* V s(y) &= 32\pi^2((\mathbb{1} + (\nabla_A)_i G_A (\nabla_A)_i) G_A) s(y) + \Delta_A k_A(y, \infty) G_A s(\infty) \\
&= 32\pi^2 G_A s(y) - 16\pi^2 (\nabla_A)_i \left[x_i, G_A\right] s(y) \\
&\quad \text{by (1.5.3), the boundary terms vanish since} \\
&\quad s \text{ has compact support} \\
&= 32\pi^2 G_A s(y) - 64 G_A s(y) - 16\pi^2 \left[x_i, G_A\right] s(y) \\
&= -32\pi^2 G_A s(y) - 16\pi^2 \left[x_i, G_A\right] s(y)
\end{aligned}$$

Hence

$$(V^* V - 4\pi^2 G_A \|x - y\|^2) s \in \ker \Delta_A.$$

Since s has compact support, the boundary terms vanish and therefore by uniqueness of solutions

$$(V^* V - 4\pi^2 G_A \|x - y\|^2) s = 0.$$

∎

We now need to show that $v^* v = \mathbb{1}$, this will follow from what we have done above, and thus prove that v is an isometry hence an isomorphism.

Proposition 1.5.8

$$v_x^* v_y = 4\pi \|x - y\|^2 k_A(x, y) \tag{1.46}$$

Proof
This follows immediately from the fact that

$$V s = \int_{\mathbb{R}^4} v_x s(x) \mathrm{d}x$$

[1] A physicist would call this taking the 2-trace by contracting the spinor parts

for $s \in L^2\Omega^0(\mathbb{R}^4; \underline{E})$ and expanding the integrals involved. ■

Now using the results from [4] that for the Hodge Laplacian, d*d, on $\mathbb{R}^4$, the kernel is given by

$$k(x,y) = \frac{1}{4\pi^2||x-y||^2} + \mathrm{O}(||x-y||^{-1})$$

and

$$\frac{k_A(x,y)}{k(x,y)} \longrightarrow \mathbb{1}$$

as $x \longrightarrow y$. From these we see that as $x \longrightarrow y$

$$v_x^* v_x = \mathbb{1}_{\underline{E}_x}.$$

Hence v is an isometry. From this, (writing $v(x)$ for v_x) we have the following result

Lemma 1.5.9 *The kernel of the Green's operator for the $\mathcal{D}$-Laplacian is given by*

$$k_{\mathcal{D}}(x,y) = \frac{v(x)^* v(y)}{4\pi^2||x-y||^2} \tag{1.47}$$

Proof
Set $w(x) = \frac{v(y)}{||x-y||^2}$ and $\varpi = vv^* = \mathbb{1}_{\underline{\mathbb{C}^k \otimes S^- \oplus E}} - \mathcal{R}^*\mathcal{F}\mathcal{R}$ by (1.33) and denote differentiation by subscripts. We need a few smaller results first.

Claim 1.5.10 $v^*\varpi_\mu = -v^*\iota q_\mu \mathcal{F}\mathcal{R}$.

Proof
(of Claim (1.5.10)) This follows almost immediately from (1.33),(1.35) and the fact that $v^*\mathcal{R}^* = 0$. ■

Similarly $\varpi\varpi_\mu = -\varpi\mathcal{R}_\mu\mathcal{F}\mathcal{R}$.

Claim 1.5.11 $v^*\varpi = v^*$.

Proof
(of Claim(1.5.11)) This is immediate from the definition of ϖ and (1.32). ■

Claim 1.5.12 $\mathcal{R}\varpi = 0$.

Proof
(of Claim (1.5.12)) This is immediate because ϖ projects onto the kernel of $\mathcal{R}$. ■

Claim 1.5.13 $v^*\varpi_\mu\varpi = 0$.

Proof
(of Claim (1.5.13)) Immediate from (1.5.12). ■

Now we look at how the $\mathcal{D}$- Laplacian affects $v(x)^*w(x)$.

$$\begin{aligned}
-\mathcal{D}_i\mathcal{D}_i v(x)^*w(x) &= v^*\frac{\partial}{\partial x_\mu}v(x)v(x)^*\frac{\partial}{\partial x_\mu}v(x)v(x)^*w(x) \\
&= v(x)^*\frac{\partial}{\partial x_\mu}\varpi(x)\frac{\partial}{\partial x_\mu}\varpi(x)w(x) \\
&= v(x)^*\frac{\partial}{\partial x_\mu}\varpi(x)\varpi(x)_\mu w(x) + v(x)^*\frac{\partial}{\partial x_\mu}\varpi(x)\varpi(x)w_\mu(x).
\end{aligned}$$

Expanding this we find that

$$\begin{aligned}
-\mathcal{D}_i\mathcal{D}_i v(x)^*w(x) = \; & v(x)^*\varpi(x)_\mu\varpi(x)_\mu w(x) \; + \; v(x)^*\varpi(x)\varpi(x)_{\mu\mu}w(x) \\
& + v(x)^*\varpi(x)\varpi(x)_\mu w_\mu(x) \; + \; v(x)^*\varpi(x)_\mu\varpi(x)w_\mu(x) \\
& + v(x)^*\varpi(x)\varpi(x)_\mu w_\mu(x) \; + \; v(x)^*\varpi(x)\varpi(x)w_{\mu\mu}(x).
\end{aligned}$$

However, the fourth term drops out by (1.5.13), and the last term also vanishes since w is obviously harmonic with respect to the standard Laplacian on $\mathbb{R}^4$. So we end up with

$$-\mathcal{D}_i\mathcal{D}_i v(x)^*w(x) = v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu}\right)w(x) + 2v(x)^*\varpi\varpi_\mu w_\mu(x). \tag{1.48}$$

By direct calculation we have

Claim 1.5.14

$$w_\mu(x) = -2\frac{(x-y)^\mu}{||x-y||^2}w(x).$$

Hence

$$\begin{aligned}
-\mathcal{D}_i\mathcal{D}_i v(x)^*w(x) &= v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu} - 4\varpi(x)_\mu\frac{(x-y)^\mu}{||x-y||^2}\right)w(x) \\
&= v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu} + 4\iota q_\mu\mathcal{F}(x)\mathcal{R}(x)\frac{(x-y)^\mu}{||x-y||^2}\right)w(x) \\
&\quad \text{by (1.5.10)} \\
&= v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu} + 4\iota(x-y)\frac{\mathcal{F}(x)\mathcal{R}(x)}{||x-y||^2}\right)w(x).
\end{aligned}$$

Since $\mathcal{R}(y)w(x) = 0$, we have

$$\begin{aligned}
-\mathcal{D}_i\mathcal{D}_i &= v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu} + 4\iota(x-y)\frac{\mathcal{F}(x)(\mathcal{R}(x)-\mathcal{R}(y))}{||x-y||^2}\right)w(x) \\
&= v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu} + 4\iota(x-y)\frac{\mathcal{F}(x)(x-y)^*\iota^*}{||x-y||^2}\right)w(x) \\
&= v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu} + 4\iota\mathcal{F}(x)\iota^*\right)w(x)
\end{aligned}$$

since $\mathcal{R}\mathcal{R}^*$ (and hence $\mathcal{F}$) commutes with Clifford multiplication via (1.31). Now

$$\begin{aligned}
v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu}\right) &= v(x)^*(\varpi\varpi_\mu)_\mu \qquad \text{(since } \varpi\varpi = \varpi\text{)}\\
&= -v(x)^*(\varpi\iota q_\mu \mathcal{F}\mathcal{R})_\mu\\
&= -v(x)^*\varpi_\mu \iota q_\mu \mathcal{F}\mathcal{R} - v(x)^*\iota q_\mu \mathcal{F}_\mu \mathcal{R} - v(x)^*\iota q_\mu \mathcal{F}\mathcal{R}_\mu\\
&\quad \text{(using Claim 1.5.11)}\\
&= v(x)^*\iota q_\mu \mathcal{F}\mathcal{R}\iota q_\mu \mathcal{F}\mathcal{R} + v(x)^*\iota q_\mu \mathcal{F}(\mathcal{R}\mathcal{R}^*)_\mu \mathcal{F}\mathcal{R} - v(x)^*\iota q_\mu \mathcal{F} {q^*}_\mu \iota^*\\
&= v(x)^*\iota\mathcal{F}\left(q_\mu \mathcal{R}\iota q_\mu + q_\mu(\mathcal{R}\mathcal{R}^*)_\mu\right) F\mathcal{R} - 4\iota\mathcal{F}\iota^*\\
&= v(x)^*\iota\mathcal{F}\left(2q_\mu \mathcal{R}\iota q_\mu + q_\mu {q^*}_\mu \iota^*\mathcal{R}^*\right)\mathcal{F}\mathcal{R} - 4\iota\mathcal{F}\iota^*.
\end{aligned}$$

Since we have $\mathcal{R}\iota = (T-x)^*$,

$$v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu}\right) = v(x)^*\iota\mathcal{F}\left(2q_\mu(T-x)^*q_\mu + 4(T-x)\right)\mathcal{F}\mathcal{R} - 4\iota\mathcal{F}\iota^*. \tag{1.49}$$

But, (1.40) shows that

$$q_\mu(T-x)^*q_\mu = -2(T-x) \tag{1.50}$$

hence we have corresponds to an element of the kernel of $\not{D}^*_{A\,\mathcal{S}}$ and

$$v(x)^*\left(\varpi_\mu\varpi_\mu + \varpi\varpi_{\mu\mu}\right) = -4\iota F\iota^* \tag{1.51}$$

Putting (1.51) back into our expression for $\mathcal{D}_i\mathcal{D}_i v(x)^* w(x)$, we find that this collapses to zero. Hence

$$\frac{v(x)^*v(y)}{||x-y||^2}$$

is $\mathcal{D}$-harmonic. Normalising this by a factor of $\frac{1}{4\pi^2}$, we get the desired result. ∎

Our task now is to show that these two connections are at least gauge equivalent.Actually we can do better than that. We have shown that the Green's operator for $\mathcal{D}$ coincides with the Green's operator for ∇_A. This means that on compactly supported sections of $\underline{E}$ over $\mathbb{R}^4$

$$\Delta^{\mathcal{D}} = \Delta^{\nabla}. \tag{1.52}$$

We have the following Lemma.

Lemma 1.5.15 *Let $\mathcal{E}$ be a hermitian vector bundle over $\mathbb{R}^4$ and let B and B' be two unitary connections on $\mathcal{E}$ with Laplacians Δ_B and $\Delta_{B'}$ respectively. Then $B = B'$ if and only if $\Delta_B = \Delta_{B'}$.*

Proof

Obviously if $B = B'$ then $\Delta_B = \Delta_{B'}$.

Suppose that $\Delta_B = \Delta_{B'}$. Since we are working on $\mathbb{R}^4$, we can write $\nabla_B = \mathrm{d}+B$ and $\nabla_{B'} = \mathrm{d}+B'$ globally. Then

$$\begin{aligned}\Delta_B &= -(\partial_i + B_i)(\partial_i + B_i) \\ &= -(\partial_{ii} + 2B_i\partial_i) + \mathcal{C}^\infty(\mathbb{R}^4)\text{-linear terms.}\end{aligned} \tag{1.53}$$

Similarly

$$\begin{aligned}\Delta_{B'} &= -(\partial_i + {B'}_i)(\partial_i + {B'}_i) \\ &= -(\partial_{ii} + 2{B'}_i\partial_i) + \mathcal{C}^\infty(\mathbb{R}^4)\text{-linear terms}\end{aligned} \tag{1.54}$$

Since the Laplacians are equal, we have

$$0 = \Delta_B - \Delta_{B'} = -2(B_i - {B'}_i)\partial_i + \mathcal{C}^\infty(\mathbb{R}^4)\text{-linear terms} \tag{1.55}$$

Extracting the 1st order part of this expression, we can clearly see that

$${B'}_i - B_i = 0 \tag{1.56}$$

for each $i = 1 \ldots 4$. This completes the proof. ■

Lemma (1.5.15) implies that $\mathcal{D}$ and ∇_A are the same connection. This completes the proof of the ADHM construction.

The ADHM theorem tells us that any $\mathrm{U}(n)$-vector bundle with ASD connection on S^4 can be constructed from ADHM data; it also gives an important corollary.

Corollary 1.5.16 *Any* $\mathrm{U}(n)$*-vector bundle with an ASD connection over* S^4 *can be reduced to an* $\mathrm{SU}(n)$*-vector bundle with an ASD connection over* S^4*.*

Proof

Recall that for ADHM data that the new connection is given by

$$F^\nabla_{ij} = \varpi\iota q_{[i}q^*_{j]}\mathcal{F}\iota^*\varpi$$

by (1.37) and (1.33). We calculate the trace of the curvature.

$$\begin{aligned}\mathrm{tr}F^\nabla_{ij} &= \mathrm{tr}\left(\varpi\iota q_{[i}q^*_{j]}\mathcal{F}\iota^*\varpi\right) = \mathrm{tr}\left(\iota^*\varpi\varpi\iota q_{[i}q^*_{j]}\mathcal{F}\right) \\ &\quad \text{by the fact that the trace is an invariant polynomial} \\ &= \mathrm{tr}\left(\iota^*\varpi\iota q_{[i}q^*_{j]}\mathcal{F}\right) \quad \text{since } \varpi \text{ is the projection} \\ &= \mathrm{tr}\left(\iota^*\iota q_{[i}q^*_{j]}\mathcal{F} - \iota^*\mathcal{R}^*\mathcal{F}\mathcal{R}\iota q_{[i}q^*_{j]}\mathcal{F}\right) \quad \text{by (1.33)} \\ &= 0 - \mathrm{tr}\left((T_\lambda - x_\lambda)\mathcal{F}(T_\mu - x_\mu)\mathcal{F}q_\lambda q^*_\mu q_{[i}q^*_{j]}\right).\end{aligned}$$

Since $q_{[i}q_{j]}^{*}\mathcal{F}$ is traceless and using the definitions of $\mathcal{R}$ and ι

$$\begin{aligned}
\mathrm{tr}F_{ij}^{\nabla} &= \sum_{\lambda} \mathrm{tr}\left((T_\lambda - x_\lambda)\mathcal{F}(T_\lambda - x_\lambda)\mathcal{F}q_{[i}q_{j]}^{*}\right) \\
&+ \sum_{\lambda<\mu}\left[\mathrm{tr}\left((T_\lambda - x_\lambda)\mathcal{F}(T_\mu - x_\mu)\mathcal{F}q_\lambda q_\mu^{*} q_{[i}q_{j]}^{*} + (T_\mu - x_\mu)\mathcal{F}(T_\lambda - x_\lambda)\mathcal{F}q_\lambda q_\mu^{*} q_{[i}q_{j]}^{*}\right)\right] \\
&= 0 + \sum_{\lambda<\mu} \mathrm{tr}\left(((T_\lambda - x_\lambda)\mathcal{F}(T_\mu - x_\mu)\mathcal{F} - (T_\mu - x_\mu)\mathcal{F}(T_\lambda - x_\lambda)\mathcal{F})\, q_\lambda q_\mu^{*} q_{[i}q_{j]}^{*}\right) \\
&\quad \text{by Clifford multiplcation} \\
&= \sum_{\lambda<\mu} \mathrm{tr}\left([(T_\lambda - x_\lambda)\mathcal{F}, (T_\mu - x_\mu)\mathcal{F}]\, q_\lambda q_\mu^{*} q_{[i}q_{j]}^{*}\right) \\
&= 0 \qquad\qquad \text{since the trace is an invariant polynomial.}
\end{aligned}$$

So the ADHM construction provides us with $\mathrm{SU}(n)$ instantons only. By the ADHM theorem, every $\mathrm{U}(n)$-instanton is realised from ADHM data. Hence every $\mathrm{U}(n)$-instanton over S^4 is an $\mathrm{SU}(n)$-instanton over S^4. ■

1.6 Observations

1.6.1 The Dimension of the Moduli Space

We now use the ADHM data to compute the dimension of the Moduli space of Instantons over S^4. This is just a simple dimension count.

The T_i consist of 4 self adjoint endomorphism of $\mathbb{C}^k$. Hence there are $4k^2$ of these.

P is a linear map $E \longrightarrow \mathbb{C}^k \otimes S^+$ which has real dimension $2 \times 2nk = 4nk$.

The first set of ADHM data consist of three equations in $iu(k) \otimes \mathfrak{su}(2)$, so there are $3k^2$ of these.

The second part of the ADHM data is an open condition, any small perturbation of a singular $\mathcal{R}$, ie which isn't surjective, will result in one that is. This therefore doesn't contribute anything to our dimension count.

Now, working through the arguments again, we see that gauge equivalent connections A and B will affect the data by

$$(T_B)_i = (T_A)_i \text{ for all } i \quad P_B = uP_A$$

where $u \in \mathrm{SU}(k)$. In reverse, changing the ADHM data by

$$T_i \mapsto uT_iu^{-1}, \qquad P \mapsto uPw^{-1}$$

for $u \in \mathrm{U}(k),\ w \in \mathrm{SU}(n)$ changes $\mathcal{R}$ to $u\mathcal{R}g^{-1}$, where

$$g\colon \underline{\mathbb{C}^k \otimes S^- \oplus E} \longrightarrow \underline{\mathbb{C}^k \otimes S^- \oplus E}$$
$$g = \begin{pmatrix} u \otimes \mathbb{1} & 0 \\ 0 & w \end{pmatrix}.$$

Hence the connection changes

$$\nabla_A \mapsto g \nabla_A g^{-1}.$$

We shall see that $\frac{\mathrm{U}(k)\times\mathrm{SU}(n)}{\pm\mathbb{1}}$ acts freely on irreducible ADHM data, and so the condition for Gauge equivalence counts k^2 for $\mathrm{U}(k)$ and n^2-1 for $\mathrm{SU}(n)$, since we are factoring out by a finite group.

Hence the data comprise of $4k^2 + 4nk - 3k^2 - k^2 - (n^2-1) = 4nk - n^2 + 1$ parameters and the dimension of the moduli space is $4nk - n^2 + 1$.

1.6.2 Self-dual Perturbations

We now try to analyse the situation, when we choose maps T and P which no longer satisfy the ADHM condition. We assume for the moment that the nondegeneracy condition is still satisfied, i.e. that the map

$$\mathcal{R} : \underline{\mathcal{H} \otimes S^- \oplus E} \longrightarrow \underline{\mathcal{H} \otimes S^+}$$

is still surjective. The first thing to note is that we still have a vector bundle E over $\mathbb{R}^4$ (minus at most k points where the gauge singularities occur) defined as the kernel of $\mathcal{R}$, and that the trivialisation given by v, still yields a connection on this bundle. This similarly patches together at the singularities to give a vector bundle over S^4.

However, consider the connection arising from the maps

$$T = 0, \quad P = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

It can be shown that, over $\mathbb{R}^4$, the curvature of this connection does *not* have a norm in L^2.

In fact the norm is given by

$$32\mathrm{vol}(\mathrm{S}^3) \int_\varepsilon^\infty \frac{2 + 2r^2 + r^4}{r^2(1+r^2)^4} dr$$

which diverges as $\varepsilon \longrightarrow 0$. However, since we have perturbed the data somewhat, we may split $\mathcal{F}$ into $\mathrm{r}\mathcal{F}$ which commutes with the spin structure and $\mathrm{i}\mathcal{F}$ which doesn't. That is

$$\mathcal{F} = \mathrm{r}\mathcal{F} + \mathrm{i}\mathcal{F}.$$

In which case, the curvature of the induced connection changes as follows,

$$\begin{aligned} F\left(A(T,P)\right) &= v^* \iota \mathrm{d}x \wedge \mathcal{F} \mathrm{d}x^* \iota^* v \\ &\qquad \text{from (1.37) (this still follows by construction),} \\ &= v^* \iota \mathrm{d}x \wedge \mathrm{r}\mathcal{F} \mathrm{d}x^* \iota^* v + v^* \iota \mathrm{d}x \wedge \mathrm{i}\mathcal{F} \mathrm{d}x^* \iota^* v. \end{aligned}$$

The first term is ASD, since $\mathrm{r}\mathcal{F}$ commutes with the q_i, and it can be shown using the standard spin structure on $\mathbb{R}^4$ that

$$\begin{aligned} dx \wedge i\mathcal{F}dx^* &= 2\mathcal{F}_{\mathfrak{i}} \otimes (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \\ &+ 2\mathcal{F}_{\mathfrak{j}} \otimes (dx^1 \wedge dx^3 + dx^4 \wedge dx^2) \\ &+ 2\mathcal{F}_{\mathfrak{k}} \otimes (dx^1 \wedge dx^4 + dx^2 \wedge dx^3) \end{aligned}$$

where $\mathcal{F}_{\mathfrak{i}}, \mathcal{F}_{\mathfrak{j}}, \mathcal{F}_{\mathfrak{k}} : \mathcal{H} \longrightarrow \mathcal{H}$ are the appropriate components of $\mathcal{F}$.

So, we see that the deviation from the ADHM data produces a self-dual perturbation of the connection. This destroys the conformal invariance of the integral of the norm of the curvature, and accounts for the fact that although we have a connection with finite L^2-energy on S^4 this is not equivalent to a connection on $\mathbb{R}^4$ with finite L^2-energy. So in order to work with the integrals of these connections, we must integrate over S^4, rather than $\mathbb{R}^4$.

Notice that in Lemma 1.1.3, we made no use of the fact that $\mathcal{R}\mathcal{R}^*$ commuted with the spin structure. All we needed was the fact that $\mathcal{R}$ was surjective. So although over S^4, we may not have a bundle E with anti-self dual connection with $c_2(E) = k$, we still have a bundle E over S^4 with $c_2(E) = k$ *and* a connection with a certain self-dual component measured by how far off $\mathcal{R}\mathcal{R}^*$ is from commuting with the spin structure.

There does seem to be a relationship with the Seiberg-Witten equations here in that given $\vec{\zeta} \in iu(k) \otimes \Im\mathbb{H}$, we try to find ADHM data (T, P) such that

$$\Im\left(\mathcal{R}(T,P)\mathcal{R}(T,P)^*\right) = \vec{\zeta},$$

thus giving a connection whose curvature has a self-dual part determined by $\vec{\zeta}$.

1.6.3 Other Remarks

Remark
We assumed in Section 1.5 that $\mathcal{R}$ is surjective at all points of $\mathbb{R}^4$ but have not proved that it is so. We said in Section 1.5 that $\mathcal{R}$ may fail to be surjective at at most k points in $\mathbb{R}^4$. We can thus create a connection $\mathcal{D}$ with the ADHM data away from the apparent critical points by the prescribed method, which is equal to ∇_A everywhere but this finite set,and hence on this finite set by continuity and boundedness of A.

Remark
The work by Taubes shows that on an arbitrary compact smooth 4-manifold, instantons can be constructed on the by adding connections whose supports lies on disjoint neighbourhoods. This results in local ADHM data, which may in some way extend to some form of sheaf of data on the manifold.

Remark
Our calculation of the dimension of the moduli space is rather heuristic, and there are a few

things to check. First we really need to check that the action of the gauge group is free. This is true of course on irreducible connections. We also have to check that the action of the gauge group is transverse to the hyperKähler reduction defined essentially with the moment map being the ADHM equations. This follows more or less by the free action on the irreducible points. However, the dimension we have received from the ADHM data agrees with the dimension of the moduli space calculated by index theory. We will explore the hyperKähler aspect of the ADHM construction later on.

Chapter 2

Comparing Infinite with Finite

The now well known construction of polynomial invariants by Donaldson in the '80s involves working with infinite dimensional spaces of connections. Physicists have been using these techniques also in the computation of various observables. In particular

Anselmi [1] has used BRST theory to show the existence of a link theory of two dimensional submanifolds of a four manifold.

Here we try to use techniques on S^4 involving the ADHM construction to reduce the problem to that of bundle theory of finite dimensional spaces.

2.1 Preliminaries

2.1.1 Curvature of bundles under group actions

Here, we recall the theory of vector bundles and group actions detailed in section 5.2.3 of [13]. Let $\widehat{\pi}\colon \widehat{E}\longrightarrow\widehat{Y}$ be a vector bundle, and G a Lie group whose action on $\hat{E}$ covers a free action on $\widehat{Y}$. Also let $\widehat{E}$ be endowed with a G-invariant connection $\widehat{\nabla}$. Our aim is to construct a connection ∇ on the factor bundle $E = \frac{\widehat{E}}{G}$ over $Y = \frac{\widehat{Y}}{G}$.

To do this we need a connection on the principal G-bundle $p : \widehat{Y} \longrightarrow Y$. This will enable us to lift tangent vectors on Y to $\widehat{Y}$ and compute directional derivatives. We suppose that it is given in the form of a horizontal distribution $H \subset \mathrm{T}\widehat{Y}$, with connection 1-form θ.

Any section $s \in \Omega^0(Y; E)$ comes from an invariant section $\widehat{s} \in \Omega^0(\widehat{Y}; \widehat{E})$. Therefore we can set

$$\widehat{\nabla_X s} = \widehat{\nabla}_{\widehat{X}}\widehat{s},$$

where $\widehat{X}$ is the horizontal lift of X. From the definition of Y, we know that $p^*E \cong \widehat{E}$, so we can consider the effect of the pull back $p^*\nabla$ on $\widehat{E}$. Now, the directional derivatives of $p^*\nabla$ will vanish on vertical vectors in $T\widehat{Y}$, hence we know that

$$\widehat{\nabla} = p^*\nabla + V$$

where V vanishes on horizontal vectors, and is equivariant under the G action by definition of $\widehat{\nabla}$.

It is therefore obvious that $V = \Psi\theta$ where Ψ is a linear $\mathfrak{g} \longrightarrow \mathrm{End}(\widehat{E})$

It can then be shown (and [13] do this to some extent) that

$$F(\widehat{\nabla})(\widehat{X_1}, \widehat{X_2}) = F(\nabla)(X_1, X_2) + \Psi F(\theta)(\widehat{X_1}, \widehat{X_2}), \tag{2.1}$$

allowing us to compare the curvature of $\widehat{\nabla}$ with ∇.

2.1.2 Review of the ADHM construction

Recall that we may form ASD SU(2)-connections on $\mathbb{R}^4$ (S^4) by choosing

$$(T, P) \quad \in \quad iu(k) \otimes \Lambda^1(\mathrm{T}^*\mathbb{R}^4) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^k \otimes S^+)$$

such that

$$(T \wedge T)^+ + (PP^*)_0 = 0.$$

Since, as real vector spaces

$$\begin{aligned} iu(k) \otimes \Lambda^1(\mathrm{T}^*\mathbb{R}^4) \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^k \otimes S^+) &\cong iu(k) \otimes \mathbb{H} \oplus \mathbb{C}^k \otimes_{\mathbb{R}} \mathbb{H} \\ &\cong \left(iu(k) \oplus \mathbb{C}^k\right) \otimes_{\mathbb{R}} \mathbb{H} \\ &=: \mathfrak{M}^k_{\mathbb{C}}, \end{aligned}$$

we can rewrite the ADHM condition as

$$\Im\,(T^*T + PP^*) = 0$$

where $\Im$ denotes the quaternionic imaginary part which is well defined as $\mathfrak{M}^k_{\mathbb{C}}$ is very much a "quaternification" of a real vector space. We also require for each $x \in \mathbb{R}^4$

$$\mathcal{R}_x = ((T - x\mathbb{1})^*, P) \colon \mathbb{C}^k \otimes S^- \oplus \mathbb{C}^2 \longrightarrow \mathbb{C}^k \otimes S^+$$

is surjective. Then we define a bundle $E = \ker \mathcal{R}$ and a connection given by $v^*\mathrm{d}v$, where

$$v_x = \begin{bmatrix} (x\mathbb{1} - T)^{*-1}P \\ \mathbb{1} \end{bmatrix} {\sigma_x}^{-1}$$

and

$${\sigma_x}^2 = \mathbb{1} + P^*((T - x\mathbb{1})^*(T - x))^{-1}P.$$

Since E is an SU(2)-bundle, we can simplify things a little here by converting the data into data on an Sp(1)-bundle. We can identify the fibre $\mathbb{C}^2$ with S^-, so that as complex vector spaces

$$\begin{aligned}\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^k \otimes S^+) &\cong \mathrm{Hom}(S^-, \mathbb{C}^k \otimes S^+) \\ &\cong \mathbb{C}^k \otimes S^+ \otimes S^- \\ &\cong \mathbb{C}^k \otimes (\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong \mathbb{C}^k \otimes_{\mathbb{R}} \mathbb{R}^4 \\ &\cong (\mathbb{R}^4)^k \otimes_{\mathbb{R}} \mathbb{C}.\end{aligned}$$

This means that $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^k \otimes S^+)$ may be regarded as the complexification of the real space

$$(\mathbb{R}^4)^k \cong \mathbb{H}^k.$$

So,using this conversion, we can recover ASD connections by choosing

$$(T, P) \in \odot^2(\mathbb{R}^k) \otimes \mathbb{H} \oplus \mathbb{H}^k =: \mathfrak{M}^k_{\mathbb{R}}$$

subject to the ADHM condition

$$\Im\,(T^*T + PP^*) = 0 \tag{2.2}$$

and the nondegeneracy condition,

$$\mathcal{R}_x \text{ is surjective for all } x \in \mathbb{R}^4. \tag{2.3}$$

Now from a result by Wood [30], every quaternionic matrix has a left quaternionic eigenvalue. This means that $(T, 0)$ always gives a reducible solution, i.e $(T, 0)$ does not satisfy (2.3).

We set $\mathbf{A}\,(k) \subset \mathfrak{M}^k_{\mathbb{R}}$ to be the elements that satisfy (2.2) and define $\mathbf{A}^*\,(k)$ to be the set of all elements of $\mathbf{A}\,(k)$ that satisfy the nondegeneracy condition (2.3).

Now the product Lie group $\mathrm{O}(k) \times \mathrm{Sp}(1)$ acts on $\mathbf{A}^*\,(k)$ by

$$(\alpha, \beta) : \ (T, P) \longrightarrow (\alpha T \alpha^{-1}, \alpha P \beta^{-1}).$$

Proposition 2.1.1 *The action of* Sp(1) *is free on* $\mathbf{A}^*\,(k)$.

Proof
Suppose there is $\beta \in \mathrm{Sp}(1)$ with $P\beta = P$. Since we require $(T, P) \in \mathbf{A}^*\,(k)$,we are assured that $P \neq 0$. Thus $P^\mu \beta = P^\mu$ for some nonzero component P^μ of P. Hence $\beta = 1$. ■

To show that $\mathrm{O}(k)$ acts freely on $\mathbf{A}^*\,(k)$, we need the following result.

Lemma 2.1.2 *If* $(T, P) \in \mathfrak{M}^k_{\mathbb{R}}$ *is fixed by* $u \in \mathrm{O}(k)$ *then there is a decomposition*

$$T = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix} \quad P = \begin{pmatrix} P' \\ 0 \end{pmatrix}.$$

Hence such $(T, P) \notin \mathbf{A}^*\,(k)$.

Proof
Suppose $u \in \mathrm{O}(k)$ fixes (T, P).

Then $uP = P$, and $P = P_i q_i$ for vectors $P_i \in \mathbb{R}^k$, and $uP_i = P_i$ for each i. Hence u has at least one eigenvector with eigenvalue 1. Decompose $\mathbb{R}^k$ into $V \oplus W$ where V is the maximal 1-eigenspace of u and W its orthogonal complement. Notice that $V \supset \mathrm{Span}\{P_i | i = 1...4\}$. With respect to this decomposition, we have

$$T = \begin{pmatrix} T' & T_0^\top \\ T_0 & T'' \end{pmatrix}, \quad P = \begin{pmatrix} P' \\ 0 \end{pmatrix},$$

and

$$u = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & u' \end{pmatrix},$$

for some $u' \in \mathrm{O}(k-l)$. Assume that $u \neq \mathbb{1}$ and hence $W \neq 0$. The condition that $uTu^{-1} = T$ shows us, in particular that $u'T_0 = T_0$, so the columns of T_0, T_0^μ say, satisfy

$$u'T_0^\mu = T_0^\mu$$

But u' does not have $+1$ as an eigenvalue by the decomposition. Hence $T_0^\mu = 0$ and $T_0 = 0$.

Thus we have the following decompositions for T and P with respect to this splitting.

$$T = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix} \cdot P = \begin{pmatrix} P' \\ 0 \end{pmatrix}.$$

Now by R. Wood [30], we know that T'' (hence T) has a left eigenvalue $\lambda \in \mathbb{H}$. So

$$\mathcal{F}_{(\lambda,T,P)} = \mathcal{R}_{(\lambda,T,P)}\mathcal{R}^*_{(\lambda,T,P)}$$

will not be invertible and $(T, P) \notin \mathbf{A}^*(k)$.

∎

Corollary 2.1.3 *The following are equivalent*

1. *(T, P) satisfies the nondegeneracy condition;*
2. *(T, P) has trivial stabiliser under $\mathrm{O}(k)$.*

Corollary 2.1.4 *The action of $\mathrm{O}(k)$ is free on $\mathbf{A}^*(k)$.*

We may have a small problem here. Although $\mathrm{O}(k)$ and $\mathrm{Sp}(1)$ individually act freely on $\mathbf{A}^*(k)$, the full group $\mathrm{O}(k) \times \mathrm{Sp}(1)$ doesn't act freely here. Each point is fixed by $\pm(\mathbb{1}, 1)$ meaning that the group of symmetries we require is

$$\frac{\mathrm{O}(k) \times \mathrm{Sp}(1)}{\mathbb{Z}_2}.$$

Not only that but for $k \geq 2$, we have some $(T, P) \in \mathfrak{M}^k_{\mathbb{R}}$ such that

$$gTg^{-1} = T, \quad gPq^{-1} = P$$

for certain $(g, q) \in \mathrm{O}(k) \times \mathrm{Sp}(1)$. We must check that they are not in $\mathbf{A}^*(k)$.

2.1.3 The Action of $\mathrm{O}(k) \times \mathrm{Sp}(1)$ on $\mathfrak{M}^k_{\mathbb{R}}$

Choose $\xi \in \mathrm{O}(k)$, and choose a basis of $\mathbb{R}^k$ (hence of $\mathbb{H}^k$) in which

$$\xi = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ -\mathbb{1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus for

$$T = \begin{pmatrix} T' & T_1 & T_2^{\top} \\ T_1^{\top} & T'' & T_3^{\top} \\ T_2 & T_3 & T''' \end{pmatrix}, \quad P = \begin{pmatrix} P' \\ P'' \\ P''' \end{pmatrix}$$

we find

$$[\xi, T] = \begin{pmatrix} T_1 + T_1^{\top} & T'' - T' & T_3^{\top} \\ T' - T'' & -T_1 - T_1^{\top} & -T_2^{\top} \\ T_3 & -T_2 & 0 \end{pmatrix}, \quad \xi P = \begin{pmatrix} -P'' \\ P' \\ 0 \end{pmatrix}$$

So if $[\xi, T] = 0$ and $\xi P = P\alpha$ then

$$T = \begin{pmatrix} T' & T_1 & 0 \\ -T_1 & T' & 0 \\ 0 & 0 & T''' \end{pmatrix}, \quad P = \begin{pmatrix} P' \\ P'\alpha \\ 0 \end{pmatrix}$$

with the condition that $|\alpha| = 1$. From this we can conclude, that if (T, P) satisfies the ADHM condition then it is a reducible solution, unless ξ is invertible and hence k is even. This is equivalent to viewing ξ as a complex structure on $\mathbb{R}^{2l}$. Let us take the complex point of view and look at the process for $\mathbb{C}^l$. Under this identification ξ becomes multiplication by i and

$$T = T' + iT'', P = P' + iP''.$$

Also our condition that $\xi P = P\alpha$ becomes

$$iP = P\alpha$$

or

$$-P' + iP' = P'\alpha + iP'\alpha.$$

Comparing the real and imaginary parts,

$$\begin{aligned} P'\alpha &= -P', \\ P'\alpha &= P', \end{aligned}$$

since α is considered to have only real coefficients. Hence $P' = 0$ and $P = 0$, and the solution is again reducible.

So far, we have proved that if the vector field induced by $(\xi, \alpha) \in \mathfrak{o}(k) \oplus \mathfrak{sp}(1)$ vanishes at a point (T, P) then (T, P) is a reducible solution. This in turn shows that the stabiliser of $(T, P) \in \mathbf{A}^*(k)$ must be a discrete, hence finite subgroup of $\mathrm{O}(k) \times \mathrm{Sp}(1)$ which is enough for our purposes.

2.1.4 Introducing the Moduli Spaces

Let

$$\tilde{M}_k = \frac{\mathbf{A}^*(k)}{\mathrm{O}(k)}$$

and

$$\begin{aligned} M_k &= \tilde{M}_k / \left(\frac{\mathrm{Sp}(1)}{\mathbb{Z}_2} \right) \\ &= \tilde{M}_k / \mathrm{SO}(3). \end{aligned}$$

Define the Atiyah map At: $\mathbf{A}^*(k) \longrightarrow \mathcal{A}_k$ where $\mathcal{A}_k$ is the space of connections of charge k by

$$(T, P) \dot{\mapsto} v^* \mathrm{d}v$$

as above. It is well known that any two elements of the same orbit produce gauge equivalent connections, and that M_k is diffeomorphic to $\mathcal{M}_k$, the moduli space of ASD connections of charge k[23].

Also, by considering $\tilde{M}_k = \frac{\mathbf{A}^*(k)}{\mathrm{O}(K)}$, we obtain the framed moduli space of connections. The manifold of equivalence classes of ADHM data under the action of $\mathrm{O}(k)$ is precisely the moduli space of framed instantons, $\tilde{\mathcal{M}}_k$.

Later on, we will often write $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ for M_k and $\tilde{M}_k$ when we have show that that the two spaces give rise to the same bundle data.

2.2 The case of charge 1 instantons ($k = 1$)

For $k = 1$, we are in a truly interesting position since for any

$$(T, P) \in \mathbf{A}^*(1) \subset \mathfrak{M}^1_{\mathbb{R}} = \odot^2(\mathbb{R}) \otimes \mathbb{H} \oplus \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H},$$

the ADHM condition

$$\Im(T^*T + PP^*) = 0$$

is automatically satisfied, hence $\mathbf{A}(1) = \mathfrak{M}^1_{\mathbb{R}}$.

Now we can construct a canonical bundle $\widehat{E} \longrightarrow \mathbf{A}(1) \times \mathbb{R}^4$ as follows.
For each $(T, P, x) \in \mathbf{A}(1) \times \mathbb{R}^4$, define the fibre of $\widehat{E}$ to be

$$\widehat{E}_{(T,P,x)} = \ker \mathfrak{R}_x = \ker(T - x, P) : \mathbb{H}^k \oplus \mathbb{H} \longrightarrow \mathbb{H}^k.$$

We can define a connection $\widehat{\nabla}$ on $\widehat{E}$ given at the point (T, P, x) by $\mathrm{At}(T, P)_x$. From the suggestive terminology, it is clear that we will choose $\widehat{Y} = \mathbf{A}(1) \times \mathbb{R}^4$. Since we have removed the singular points, the action of

$$\frac{\mathrm{O}(1) \times \mathrm{Sp}(1)}{\mathbb{Z}_2} = \frac{\mathbb{Z}_2 \times \mathrm{Sp}(1)}{\mathbb{Z}_2} = \mathrm{Sp}(1)$$

is free. It is also clear that by construction $\widehat{\nabla}$ is $\mathrm{Sp}(1)$ invariant.

2.2.1 The curvature of the Sp(1)-bundle

We now need to consider the principal Sp(1)-bundle

$$\mathbf{A}^*(1) \longrightarrow M_1.$$

For $(T, P) \in \mathbf{A}^*(1)$, the action of the group $\mathrm{Sp}(1)$ is

$$u : (T, P) \mapsto (T, Pu^{-1}).$$

The vertical subspace $V_{(T,P)}$ will therefore be

$$\{(0, -P\widehat{u}) | \widehat{u} \in \mathfrak{sp}(1)\}.$$

We define a horizontal subspace $H_{(T,P)}$ to be the orthogonal complement of the vertical subspace. Thus $H_{(T,P)}$ will be the space

$$\{(t, p) | P^* p \in \mathbb{R}\} = \ker(p \mapsto \Im(P^* p))\}.$$

This immediately gives us the connection 1-form

$$\theta_{(T,P)}(t, p) = -\frac{1}{|P|^2} \Im(P^* p). \tag{2.4}$$

Proposition 2.2.1 *θ is indeed a connection 1-form.*

Proof
First, θ is well defined as $P \neq 0$.

On vertical vectors

$$\begin{aligned} \theta_{(T,P)}(0, -P\widehat{u}) &= -\frac{1}{|P|^2} \Im(-P^* P\widehat{u}) \\ &= \Im(\widehat{u}) \\ &= \widehat{u}. \end{aligned}$$

Also, for equivariance,

$$\begin{aligned}
\theta_{(T,Pu^{-1})}(t,pu^{-1}) &= -\frac{1}{|Pu|^2}\Im(uP^*pu^{-1}) \\
&= -\frac{1}{|P|^2}u\Im(P^*p)u^{-1} \\
&= \mathrm{ad}(u)\theta_{(T,P)}(t,p).
\end{aligned}$$

■

Using local coordinates (T,P), this connection 1-form can be written

$$\theta = -\frac{1}{|P|^2}\Im(P^*\mathrm{d}P),$$

so the curvature form restricted to the horizontal space is given by

$$\begin{aligned}
F(\theta)\Big|_{\ker\theta} &= \mathrm{d}\theta\Big|_{\ker\theta} \\
&= -\frac{1}{|P|^2}\mathrm{d}P^*\wedge\mathrm{d}P\Big|_{\ker\theta} - \frac{1}{|P|^2}\mathrm{d}|P|^2\wedge\theta\Big|_{\ker\theta} \\
&= -\frac{1}{|P|^2}\mathrm{d}P^*\wedge\mathrm{d}P\Big|_{\ker\theta}.
\end{aligned}$$

Now $(t,p)\in\mathbf{A}\,(1)$ is horizontal if and only if $p=\lambda P$ by the definition of θ in (2.4). Hence for a horizontal vector $(X,t,\lambda P)\in\mathbb{R}^4\times H_{(T,P)}$

$$\mathrm{d}P((X,t,\lambda P)) = \lambda P,$$

and from this we can see that

$$\mathrm{d}P\Big|_{\ker\theta} = \frac{\delta|P|^2}{2|P|^2}P = \frac{\delta|P|}{|P|}\Big|_{\ker\theta}P. \tag{2.5}$$

As a result

$$F(\theta)\Big|_{\ker\theta} = -\delta|P|\wedge\delta|P|\Big|_{\ker\theta} = 0.$$

This means that the formula (2.1) becomes

$$F(\nabla)(X_1,X_2) = F(\widehat{\nabla})(\widehat{X_1},\widehat{X_2}).$$

2.2.2 The Curvature of the Universal Bundle

Recall that the Atiyah map was defined by

$$\begin{aligned}
\mathrm{At}\colon \mathbf{A}\,(1) &\longrightarrow \mathcal{A}_k \\
(T,P) &\mapsto v(T,P)^*\mathrm{d}v(T,P).
\end{aligned}$$

Explicitly for $k=1$, $\mathrm{At}(T,P)$ is given by $v(T,P)^*\mathrm{d}v(T,P)$ where

$$v(T,P) = \frac{|x-T|}{\sqrt{|P|^2+|x-T|^2}}\begin{bmatrix} \frac{(x-T)}{|x-T|^2}P \\ 1 \end{bmatrix}.$$

Thus $\widehat{\nabla}$ is given at the point (T,P,x) by $v(T,P)^*_x\widehat{\mathrm{d}}v(T,P)_x$, where $\widehat{\mathrm{d}}$ denotes the de Rham differential not only with respect to x but also with respect to the quaternions T and P. We will also denote by δ the de Rham differential in the ADHM space. Also, set $\tilde{x} = x - T$

Expanding this calculation and setting

$$\Delta = \sqrt{|P|^2+|\tilde{x}|^2},$$

we find that the connection 1-form of $\widehat{\nabla}$ with respect to the global trivialisation of $\widehat{E}$ is

$$v(T,P)^*_x\widehat{\mathrm{d}}v(T,P)_x = \frac{1}{\Delta^2}\Im\left(P^*\mathrm{d}P - \frac{1}{|\tilde{x}|^2}P^*\widehat{\mathrm{d}}\tilde{x}^*\tilde{x}P\right). \tag{2.6}$$

We compute the connection in the obvious way, i.e.

$$F(A) = \mathrm{d}A + A\wedge A.$$

From this we get

$$\begin{aligned} F(\widehat{\nabla}) = \tfrac{1}{\Delta^4}\Big[&|\tilde{x}|^2\mathrm{d}P^*\wedge\mathrm{d}P + \tfrac{1}{|\tilde{x}|^2}P^*\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}\wedge\widehat{\mathrm{d}}\tilde{x}^*\tilde{x}P \\ &-P^*\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}\wedge\mathrm{d}P - \mathrm{d}P^*\wedge\widehat{\mathrm{d}}\tilde{x}^*\tilde{x}P\Big]. \end{aligned} \tag{2.7}$$

Now, we are interested in "factoring" out $\widehat{\nabla}$ by $\mathrm{Sp}(1)$. That is, we want only to consider the horizontal lifts of tangent vectors and vector fields on M_1 and how $F(\widehat{\nabla})$ behaves when restricted to these.

So by (2.5), we have

$$\begin{aligned} &F(\widehat{\nabla})\Big|_{\ker\theta} \\ =\ &\frac{1}{\Delta^4}\left[|\tilde{x}|^2\delta|P|\wedge\delta|P| + \frac{1}{|\tilde{x}^2|}P^*\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}\wedge\widehat{\mathrm{d}}\tilde{x}^*\tilde{x}P - 2\Im(P\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}\wedge\frac{\delta|P|}{|P|}P)\right] \\ =\ &\frac{1}{\Delta^4}\left[\frac{1}{|\tilde{x}^2|}P^*\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}\wedge\widehat{\mathrm{d}}\tilde{x}^*\tilde{x}P + 2\frac{\delta|P|}{|P|}\wedge\Im(P\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}P)\right]. \end{aligned}$$

If we set $|P| = \rho$, then we find that when restricted to the horizontal space

$$F(\widehat{\nabla}) = \frac{1}{\Delta^4}\left[\frac{1}{|\tilde{x}^2|}P^*\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}\wedge\widehat{\mathrm{d}}\tilde{x}^*\tilde{x}P + 2\frac{\delta\rho}{\rho}\wedge\Im(P\tilde{x}^*\widehat{\mathrm{d}}\tilde{x}P)\right]. \tag{2.8}$$

This is very similar to Anselmi's formula in [1]. In fact if we assume, (as he effectively does) that P is real and positive under part of a gauge fixing condition, then we have his answer scaled by a factor of $\frac{1}{2}$.

2.2.3 Calculating the μ map

Our next goal is to compute a form representing the second Chern class of E,$c_2(E)$. This will be the same as the second Chern character $\mathrm{ch}_2(E)$ since the group is $SU(2) \cong Sp(1)$ and hence $c_1(E) = 0$. We may obtain a representative of the cohomology class of $c_2(E)$ given by

$$\frac{1}{4\pi^2}\mathrm{tr}\,(F(\nabla) \wedge F(\nabla)).$$

By expanding this we find that the desired form is

$$c = \frac{6\rho^4}{\Delta^8\pi^2}\widehat{\mathrm{d}}\tilde{x}_1 \wedge \widehat{\mathrm{d}}\tilde{x}_2 \wedge \widehat{\mathrm{d}}\tilde{x}_3 \wedge \widehat{\mathrm{d}}\tilde{x}_4 - \frac{6\rho^3}{\Delta^8\pi^2}\delta\rho \wedge \sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4 \tag{2.9}$$

where $\tilde{x}_i = x_i - T_i$.

Now, it can be checked that

$$c = \widehat{\mathrm{d}}\left(\frac{1}{2\pi^2}\frac{(|\tilde{x}|^2 + 3\rho^2)}{(|\tilde{x}|^2 + \rho^2)^3}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4\right).$$

so that for any d-dimensional submanifold Σ of $\mathbb{R}^4$ with Poincaré dual α and $(4-d)$-dimensional submanifold Ξ of M

$$\begin{aligned}
\int_{\Xi\times\Sigma} c &= \int_{\Xi\times\mathbb{R}^4} c \wedge \alpha \\
&= \int_{(\partial M\cap\Xi)\times\mathbb{R}^4}\left(\frac{1}{2\pi^2}\frac{(|\tilde{x}|^2 + 3\rho^2)}{(|\tilde{x}|^2 + \rho^2)^3}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4\right) \wedge \alpha \\
&= \frac{1}{2\pi^2}\int_{(\Xi\cap\{\rho=0\})\times\mathbb{R}^4}\frac{1}{|\tilde{x}|^4}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4 \wedge \alpha \\
&= \frac{1}{2\pi^2}\int_{(\Xi\cap\{\rho=0\})\times\Sigma}\frac{1}{|\tilde{x}|^4}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4.
\end{aligned}$$

The right hand side above can be seen to be the Gauß formula for the linking number of Σ with $\Xi \cap \{P = 0\}$ regarded as a 3-dimensional submanifold of $\mathbb{R}^4$.

2.2.4 The Donaldson Polynomials in the case $k = 1$

We now have to assess the consequences of this for the Donaldson Polynomials. Let $\Sigma_1, \ldots, \Sigma_d$ be d submanifolds of $\mathbb{R}^4$. We may form the Donaldson μ class

$$\mu_{Don}(\Sigma_i) = \int_{\Sigma_i} c$$

and from this the Donaldson polynomial

$$\begin{aligned}
\mathrm{Don}_1(\Sigma_1, \ldots, \Sigma_d) &= \int_{\mathcal{M}_1}\mu_{Don}(\Sigma_1)\ldots\mu_{Don}(\Sigma_d) \\
&= \int_{\mathcal{M}_1\times\Sigma_1\times\ldots\times\Sigma_d} c_1 \wedge \ldots \wedge c_d
\end{aligned}$$

where c_i is c restricted to Σ_i.

Now let

$$f(x,T,\rho)=\frac{1}{2\pi^2}\frac{\left(|\tilde{x}|^2+3\rho^2\right)}{\left(|\tilde{x}|^2+\rho^2\right)^3}$$

so that

$$c=\widehat{\mathrm{d}}\left(f(x,T,P)\alpha(x,T,P)\right)$$

where

$$\alpha(x,T,P)=\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1\ldots\wedge\hat{i}\wedge\ldots\wedge\widehat{\mathrm{d}}\tilde{x}_4.$$

Then

$$\begin{aligned}
&\mathrm{Don}_1(\Sigma_1,\ldots,\Sigma_d)\\
=&\int_{\mathcal{M}_1\times\Sigma_1\times\ldots\times\Sigma_d} c_1\wedge\ldots\wedge c_d\\
=&\int_{\mathcal{M}_1\times\Sigma_1\times\ldots\times\Sigma_d}\widehat{\mathrm{d}}(f(x^1,T,P)\alpha(x^1,T,P))\wedge\ldots\wedge\widehat{\mathrm{d}}(f(x^d,T,P)\alpha(x^d,T,P))\\
=&\int_{\mathcal{M}_1\times\Sigma_1\times\ldots\times\Sigma_d}\widehat{\mathrm{d}}\left(f(x^1,T,P)\alpha(x^1,T,P)\wedge\ldots\wedge\widehat{\mathrm{d}}(f(x^d,T,P)\alpha(x^d,T,P))\right)\\
=&\int_{\{\rho=0\}\times\Sigma_1\times\ldots\times\Sigma_d} f(x^1,T,P)\alpha(x^1,T,P)\wedge\ldots\wedge\widehat{\mathrm{d}}(f(x^d,T,P)\alpha(x^d,T,P))\\
=&\frac{1}{2\pi^2}\lim_{\rho\longrightarrow 0}\int_{\{(T,\rho)\}\times\Sigma_1\times\ldots\times\Sigma_d}\frac{(3\rho^2+|x^1-T|^2)}{(\rho^2+|x^1-T|^2)^3}\alpha(x^1,T,0)\wedge\\
&\prod_{l=1}^{d}\left(\frac{6\rho^4}{(\rho^2+|x^l-T|^2)^4}\widehat{\mathrm{d}}(x_1^l-T_1)\wedge\ldots\widehat{\mathrm{d}}(x_4^l-T_4)\right)
\end{aligned}$$

using the formula (2.9) for c.

Now, since we have a singularity when $x=T$ and $\rho=0$ in the formula

$$\frac{6\rho^4}{(\rho^2+|x-T|^2)^4}$$

we have to be a bit careful with limits. Now, a calculation shows that

$$\lim_{\rho\longrightarrow 0}\frac{6\rho^4}{(\rho^2+|x-T|^2)^4}=\frac{1}{2}\mathrm{vol}\,\mathrm{S}^3\delta(x-T)$$

and provided $x^1,\ldots,x^d$ are distinct points in $\mathbb{R}^4$

$$\lim_{\rho\longrightarrow 0}\prod_{l=1}^{d}\frac{6\rho^4}{(\rho^2+|x^l-T|^2)^4}=\prod_{l=1}^{d}\frac{1}{2}\mathrm{vol}\,\mathrm{S}^3\delta(x^l-T).$$

For $d=2$, we have

$$\mathrm{Don}_1(\Sigma_1,\Sigma_2)=\frac{\mathrm{vol}\,\mathrm{S}^3}{2}\int_{\Sigma_1}\int_{\Sigma_2}\frac{1}{|x^1-x^2|^2}\alpha(x^1,x^2,0)$$

which is a constant multiple of the linking number of Σ_1 with Σ_2.

For $d > 2$, the situation is much more complicated. A discussion of this can be found in another article by Anselmi, [2].

2.3 The case for general k

We hope to proceed analogously with the general case, though we would expect this to be significantly more complicated. The purpose of this calculation is to show that the induced forms in both the infinite dimensional and finite dimensional constructions are exactly the same. To do this, we shall adopt the following procedure.

We will compute the curvatures for each version of the framed moduli space, which differs slightly from the reduced space model given in [13]. We will then show that in fact the curvatures are exactly the same which will give us the result that in the framed situation there is no difference between the representatives of the μ class.

2.3.1 The Universal Bundle

As before we define the bundle $\widehat{E}$ fibrewise over $\mathbb{R}^4 \times \mathbf{A}^*(k)$

$$\widehat{E}_{(x,T,P)} = \ker \mathcal{R}_{(x,T,P)} = \ker \begin{pmatrix} -\tilde{x}^* & P \end{pmatrix}$$

where $\tilde{x} = x\mathbb{1} - T$ regarding $x \in \mathbb{R}^4$ as a quaternion. We have an obvious action of both $\mathrm{O}(k)$ and $\mathrm{Sp}(1)$ on $\widehat{E}$ when we regard it as a subbundle of the trivial bundle $\underline{\mathcal{H} \otimes S^- \oplus S^+}$. We have

$$(g, \alpha) : \begin{pmatrix} s \\ e \end{pmatrix} \mapsto \begin{pmatrix} gs \\ \alpha e \end{pmatrix}.$$

We remark here that $\mathrm{Sp}(1)$ is acting here as elements of $\mathrm{End}(S^+)$ and in that sense acts trivially on negative spinors. Since we have a group acting on the total space $\widehat{E}$ here, we need to examine how this relates to the action over $\mathbb{R}^4 \times \mathbf{A}^*(k)$. First notice that if

$$\begin{pmatrix} s \\ e \end{pmatrix} \in \widehat{E}_{(x,T,P)} = \ker \begin{pmatrix} -\tilde{x}^* & P \end{pmatrix}$$

then

$$\begin{aligned} (g,\alpha)\begin{pmatrix} s \\ e \end{pmatrix} = \begin{pmatrix} gs \\ \alpha e \end{pmatrix} &\in \ker \begin{pmatrix} -\tilde{x}^* g^{-1} & P\alpha^{-1} \end{pmatrix} \\ &= \ker \begin{pmatrix} -g\tilde{x}g^{-1} & gP\alpha^{-1} \end{pmatrix} \\ &= \ker \mathcal{R}_{(x, gTg^{-1}, gP\alpha^{-1})} \\ &= \widehat{E}_{(x, gTg^{-1}, gP\alpha^{-1})}. \end{aligned}$$

From this we see that for each $(g, \alpha) \in \mathrm{O}(k) \times \mathrm{Sp}(1)$ and $e \in \widehat{E}$, we have

$$\pi_{\widehat{E}}\left((g,\alpha)\cdot e\right) = (g,\alpha)\cdot \pi_{\widehat{E}}(e)$$

where $\pi_{\widehat{E}}$ is the bundle projection. Hence $\widehat{E}$ is an equivariant bundle.

Recall that, since $\mathrm{O}(k) \times \mathrm{Sp}(1)$ acts on $\mathbf{A}^*(k)$ such that each point is fixed by $\pm(\mathbb{1}, \mathbb{1})$ we restrict to the group

$$\frac{\mathrm{O}(k) \times \mathrm{Sp}(1)}{\mathbb{Z}_2}.$$

Recall that for $k = 1$ we defined the connection $\widehat{\nabla}$ by

$$\widehat{\nabla} = \varpi \widehat{\mathrm{d}} \varpi$$

where $\varpi(x, T.P) : \mathcal{H} \otimes S^- \oplus S^+ \longrightarrow \mathcal{H} \otimes S^- \oplus S^+$ is the orthogonal projection onto $\ker \mathcal{R}_{(x,T,P)}$. Since ϖ is defined by a metric preserved by $\mathrm{O}(k) \times \mathrm{Sp}(1)$, it must therefore be an invariant connection $\widehat{\nabla}$ on $\widehat{E}$ allowing us to apply the theory set up in 2.1.1. In the trivialisation provided by v, the connection becomes

$$\widehat{\nabla} = v^* \widehat{\mathrm{d}} v.$$

2.3.2 The Action of $\mathrm{O}(k)$

We try to find a connection on $\mathbf{A}(k)$ using the same method as for the $k = 1$ case. At $(T, P) \in \mathbf{A}(k)$, the tangent space to $\mathbf{A}(k)$ is

$$\{(t, p) \in \mathfrak{M}^k_{\mathbb{R}} | \Im(T^*t + t^*T + Pp^* + pP^*) = 0\}.$$

Since $\mathrm{O}(k)$ acts on $\mathbf{A}^*(k)$ by

$$\alpha : (T, P) \mapsto (\alpha T \alpha^{-1}, \alpha P),$$

the vertical subspace is given by

$$V_{(T,P)} = \{([\alpha, T], \alpha P) | \alpha \in \mathfrak{o}(k)\}.$$

Now $\mathfrak{M}^k_{\mathbb{R}}$ is a Euclidean space with Euclidean metric given by

$$\langle (t, p), (t', p') \rangle = \frac{1}{2} \Re \mathrm{tr}_{\mathbb{R}}(t^* t') + \Re(p^* p') \tag{2.10}$$

where $\mathrm{tr}_{\mathbb{R}} : \odot^2(\mathbb{R}^k) \otimes \mathbb{H} \longrightarrow \mathbb{H}$ is given by

$$\mathrm{tr}_{\mathbb{R}}(t^i \otimes q_i) = \mathrm{tr}(t^i) q_i.$$

Proposition 2.3.1 *This inner product is invariant under the action of* $\mathrm{O}(k)$.

Proof
The tangent representation of $\mathrm{O}(k)$ on $\mathbf{A}(k)$ is

$$\alpha : (t, p) \mapsto (\alpha t \alpha^{-1}, \alpha p),$$

so

$$\begin{aligned}
\langle\alpha\cdot(t,p),\alpha\cdot(t',p')\rangle &= \frac{1}{2}\Re\mathrm{tr}_{\mathbb{R}}(\alpha t^*\alpha^\top\alpha t'\alpha^\top)+\Re(p^*\alpha^\top\alpha p')\\
&= \frac{1}{2}\Re\mathrm{tr}_{\mathbb{R}}(\alpha t^* t'\alpha^\top)+\Re(p^*p')\\
&= \frac{1}{2}\Re\mathrm{tr}_{\mathbb{R}}(t^*t')+\Re(p^*p').
\end{aligned}$$

■

This metric restricts to a metric on $\mathbf{TA}^*(k)$. Proceeding as before, we define the equivariant horizontal subspace

$$H_{(T,P)} = (V_{(T,P)})^\perp.$$

We would like a more explicit description of a horizontal vector in order to build a connection 1-form. A pair (t,p) lies in $H_{(T,P)}$ if

$$0 = \langle (t,p),([\alpha,T],\alpha P)\rangle \tag{2.11}$$

for all $\alpha\in\mathfrak{o}(k)$.

Using the summation convention on quaternionic indices i,j,

$$\begin{aligned}
0 &= \langle (t,p),([\alpha,T],\alpha P)\rangle = \frac{1}{2}\Re\mathrm{tr}_{\mathbb{R}}(t^*[\alpha,T])+\Re(p^*\alpha P)\\
&= \frac{1}{2}\mathrm{tr}\Re(t^*\alpha T-t^*T\alpha)+\Re(p^*\alpha P) = \frac{1}{2}\mathrm{tr}(t_i\alpha T_i - t_iT_i\alpha+2p_i\alpha P_i^\top)\\
&= \frac{1}{2}\mathrm{tr}(T_it_i\alpha - t_iT_i\alpha+2p_i\alpha P_i^\top) = \frac{1}{2}\mathrm{tr}\left((T_it_i\alpha)^\top\right)-\frac{1}{2}\mathrm{tr}(t_iT_i\alpha+2p_i\alpha P_i^\top)\\
&= -\mathrm{tr}(t_iT_i\alpha+p_i\alpha P_i^\top) = \Re\mathrm{tr}_{\mathbb{R}}\left((-t^*T-pP^*)\alpha\right).
\end{aligned}$$

This means that

$$(t^*T+pP^*)\in\odot^2(\mathbb{R}^k)\otimes\mathbb{H}. \tag{2.12}$$

However, since $(t,p)\in \mathrm{T}_{(T,P)}\mathbf{A}^*(k)$

$$\Im(T^*t+t^*T+Pp^*+pP^*)=0. \tag{2.13}$$

Together (2.12) and (2.13) imply that

$$\Re\begin{bmatrix} t\\ p^*\end{bmatrix}\in\odot^2(\mathbb{R}^k)\otimes\mathbb{H}. \tag{2.14}$$

It therefore makes sense to consider the 1-form $\tilde{\theta}^1$ given by

$$\begin{aligned}
\tilde{\theta}^1_{(T,P)}(t,p) &= \frac{1}{2}\Re\left(((T^*t+Pp^*)-(T^*t+Pp^*)^\top)\right)\\
&= ([T_i,t_i]+P_ip_i^\top - p_iP_i^\top),
\end{aligned}$$

that is

$$\tilde{\theta}^1 = \Re(\mathcal{R}\delta\mathcal{R}^* - \delta\mathcal{R}\mathcal{R}^*).$$

By Proposition 2.3.1, $\tilde{\theta}^1$ is an equivariant 1-form but does not behave as the identity on vertical vectors. Indeed, define $\Phi\colon \mathfrak{o}(k)\longrightarrow\mathfrak{o}(k)$ by

$$\begin{aligned}\Phi(\hat{\alpha}) &= \tilde{\theta}^1_{(T,P)}([\hat{\alpha},T],\hat{\alpha}P)\\ &= \Re\left(2T^*\hat{\alpha}T - \hat{\alpha}T^*T - T^*T\hat{\alpha} - \hat{\alpha}PP^* - PP^*\hat{\alpha}\right)\\ &= \Re\left(2\tilde{x}^*\hat{\alpha}\tilde{x} - \hat{\alpha}\mathcal{R}\mathcal{R}^* - \mathcal{R}\mathcal{R}^*\hat{\alpha}\right).\end{aligned}$$

This is clearly not the identity mapping for generic (T,P), but we do have the following result.

Lemma 2.3.2 Φ *is invertible.*

Proof

It is enough to show that $\langle\hat{\alpha},\Phi(\hat{\alpha})\rangle_{\mathfrak{o}(k)} \neq 0$ for each $\alpha \in \mathfrak{o}(k)\backslash\{0\}$. We have

$$\begin{aligned}\langle\hat{\alpha},\Phi(\hat{\alpha})\rangle_{\mathfrak{o}(k)} &= -\mathrm{tr}\left(\hat{\alpha}\Phi(\hat{\alpha})\right)\\ &= -\mathrm{tr}\Re\left(2\hat{\alpha}T^*\hat{\alpha}T - \hat{\alpha}\hat{\alpha}T^*T - \hat{\alpha}T^*T\hat{\alpha} - \hat{\alpha}\hat{\alpha}PP^* - \hat{\alpha}PP^*\hat{\alpha}\right)\\ &= \mathrm{tr}\Re\left(2(T\hat{\alpha})^*(\hat{\alpha}T) - (T\hat{\alpha})^*(T\hat{\alpha}) - (\hat{\alpha}T)^*(\hat{\alpha}T)\right) - 2\Re\left((P\hat{\alpha})^*(P\hat{\alpha})\right)\\ &= -\|[\hat{\alpha},T]\|^2_{\mathrm{End}(\mathbb{R}^k)\otimes\mathbb{H}} - 2\|\hat{\alpha}P\|^2_{\mathrm{End}(\mathbb{R}^k)\otimes\mathbb{H}}.\end{aligned}$$

This vanishes iff $[\hat{\alpha},T] = 0$ and $\hat{\alpha}P = 0$, that is, iff $\exp(t\hat{\alpha})T\exp(-t\hat{\alpha}) = T$ and $\exp(t\hat{\alpha})P = P$ for each $t \in \mathbb{R}$. This cannot happen unless $\hat{\alpha} = 0$, since the action of $\mathrm{O}(k)$ is free by Lemma 2.1.4. ∎

Hence we get a connection 1-form θ^1 given by

$$\theta^1 = \Phi^{-1}\circ\tilde{\theta}^1.$$

Now we have a representation of the Lie algebra $\mathfrak{o}(k)$ on $\widehat{E}$ given by

$$\Psi_1 : \alpha \mapsto v^*\iota\alpha\iota^*v = v^*\begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix}v.$$

We are primarily interested in finding the curvature of this restricted to the horizontal space. As a result, the quadratic term $\theta^1\wedge\theta^1$ vanishes on this space and so

$$F(\theta^1)|_H \;=\; \widehat{\mathrm{d}}\theta^1|_H.$$

2.3.3 The Curvature of the Universal Bundle

On $\mathbb{R}^4$ we have a formula for the trivialisation of an instanton given by ADHM data (T,P) away from the points $x \in \mathbb{R}^4$ such that there is a vector $U \in \mathbb{R}^k$ such that $T_iU = x_iU$ for all i. Namely

$$v_x = \begin{bmatrix}(x\mathbb{1} - T)^{-1^*}P\\ \mathbb{1}\end{bmatrix}\sigma(x)^{-1}$$

where

$$\sigma(x)^2 = \mathbb{1} + P^*((x\mathbb{1} - T)^*(x\mathbb{1} - T))^{-1}P$$

and is well defined since the right hand side is a self-adjoint matrix with strictly positive eigenvalues. Hence we can trivialise $\widehat{E}$ using

$$v_{(x,T,P)} = \begin{bmatrix} {(x\mathbb{1} - T)^{-1}}^* P \\ \mathbb{1} \end{bmatrix} \sigma(x,T,P)^{-1}.$$

On $\widehat{E}$, we have the connection $\widehat{\nabla}$ given at (x, T, P) by

$$\widehat{\nabla} = v^*_{(x,T,P)} \widehat{\mathrm{d}} v_{(x,T,P)},$$

the curvature of which is given by

$$F(\widehat{\nabla}) \;=\; v^* \widehat{\mathrm{d}} v v^* \widehat{\mathrm{d}} v \;=\; -v^* \widehat{\mathrm{d}} \mathcal{R}^* \wedge \mathcal{F} \mathcal{R} \widehat{\mathrm{d}} v,$$

where

$$\mathcal{F} = (\mathcal{R}\mathcal{R}^*)^{-1}$$

which exists by surjectivity of $\mathcal{R}$. By using similar methods to those in the proof of (1.36) of 1.5.2, we can use the fact that

$$\mathcal{R} v = 0$$

to prove that

$$\begin{aligned} F(\widehat{\nabla}) &= v^* (\widehat{\mathrm{d}} \mathcal{R}^*) \mathcal{F} \wedge (\widehat{\mathrm{d}} \mathcal{R}) v \\ &= v^* \begin{bmatrix} -\mathrm{d}\tilde{x} \\ \mathrm{d}P^* \end{bmatrix} \mathcal{F} \wedge \begin{bmatrix} -\mathrm{d}\tilde{x}^* & \mathrm{d}P \end{bmatrix} v. \end{aligned}$$

So, using the formula (2.1), we find that

$$\begin{aligned} F(\nabla) &= F(\widehat{\nabla})|_H - \Psi_1 F(\theta^1)|_H \\ &= F(\widehat{\nabla})|_H - \Psi_1 \mathrm{d}\theta^1|_H. \end{aligned}$$

2.3.4 Recalling the Infinite Dimensional Construction

The expression that we have for the curvature in the finite dimensional case given in 2.3.3 is rather involved and the method of computing the μ-map is rather fraught with danger of algebraic inaccuracy. Instead we compare the curvature obtained by the ADHM construction as before with the infinite dimensional case given in section 5.2 of [13]. First we recall this construction.

The "infinite dimensional" curvature $F(\nabla^\infty)$ is obtained by considering the universal adjoint bundle over $\tilde{\mathcal{B}}(k) \times \mathbb{R}^4$, the product of the moduli space of irreducible connections on the bundle of charge k with Euclidean space.

Set $\widehat{E}^{\infty}$ to be the pull back of E over $\mathcal{A}^*(k) \times \mathbb{R}^4$ by projection on to the second factor and define $\widehat{\nabla}^{\infty}$ to be the connection trivial in the $\mathcal{A}^*(k)$ direction and tautological in the $\mathbb{R}^4$ direction, ie at the point (A, x)

$$\widehat{\nabla}^{\infty} = \mathrm{proj}_2^* \nabla_A.$$

We consider the quotient of this under the action of $\mathcal{G}_0$, but a quotient SU(2) bundle might not exist. We consider instead the bundle of Lie algebras $\mathrm{Ad}\widetilde{\mathbb{P}}$ given by

$$\mathrm{Ad}\widetilde{\mathbb{P}} = \frac{\mathrm{Ad}\widehat{E}^{\infty}}{\mathcal{G}_0}.$$

We have the principal bundle $\mathcal{A}^*(k) \longrightarrow \mathcal{B}^*(k)$ and would like to find a connection θ^{∞} on this bundle.

Recall that for the bundle $E \longrightarrow \mathbb{R}^4$ we have the complex

$$0 \longrightarrow V_0 \longrightarrow \Omega^0(\mathbb{R}^4; \mathrm{Ad}E) \xrightarrow{\mathrm{d}_A} \Omega^1(\mathbb{R}^4; \mathrm{Ad}E) \xrightarrow{\mathrm{d}_A^+} \Omega^2_+(\mathbb{R}^4; \mathrm{Ad}E) \longrightarrow 0 \tag{2.15}$$

where V_0 is the set of covariant constants given in 1.2. Recall also that these covariant constants occur as harmonic sections which evaluate at infinity. Hence V_0 is a 3-dimensional space and represents the complement of $\mathcal{G}_0$ within $\mathcal{G}$.

This sequence, although not exact, has the property that $\mathrm{im}\mathrm{d}_A \subset \ker \mathrm{d}_A^+$.

We alter the complex to

$$0 \longrightarrow \Omega^0(\mathbb{R}^4; \mathrm{Ad}E) \xrightarrow{\mathcal{P}_A} \Omega^1(\mathbb{R}^4; \mathrm{Ad}E) \oplus V_0 \xrightarrow{\mathcal{Q}_A} \Omega^2_+(\mathbb{R}^4; \mathrm{Ad}E) \longrightarrow 0 \tag{2.16}$$

where

$$\begin{aligned} \mathcal{P}_A(a) &= (\mathrm{d}_A a, s_{a(\infty)}) \\ \mathcal{Q}_A(b, s_e) &= \mathrm{d}_A^+ b. \end{aligned}$$

We have a map

$$\mathcal{S}_A : \Omega^1(\mathbb{R}^4; \mathrm{Ad}E) \oplus V_0 \longrightarrow \Omega^0(\mathbb{R}^4; \mathrm{Ad}E)$$
$$\mathcal{S}_A(b, s_e) = \mathrm{d}_A^* b + s_e.$$

We can see that this is injective, since $\mathrm{im}\, \mathrm{d}_A^* = (\ker \mathrm{d}_A)^{\perp}$, and

$$\mathcal{S}_A \mathcal{P}_A(a) = \mathrm{d}_A^* \mathrm{d}_A a + s_{a(\infty)}$$

is invertible with inverse

$$a \mapsto G_A \Pi a + s_{a(\infty)}.$$

where Π is the projection

$$\Pi(a) = a - s_{a(\infty)}$$

using the notation in 1.2.

So the Zariski tangent space to the framed moduli space of ASD connections on E, will be the first homology of (2.16). The horizontal subspace at A is therefore isomorphic to the direct sum $\ker d_A^* \oplus V_0$, and a horizontal tangent vector is a pair $(a, s_e) \in \ker \mathrm{d}^* \oplus V_0$.

The connection on this bundle can therefore be given by

$$\theta^{\infty}{}_A(a, s_e) = -\Pi G_A \mathrm{d}_A^* a,$$

where $a \in \mathrm{T}_A\mathcal{A}^*(k) \cong \Omega^1(\mathbb{R}^4; \mathrm{Ad}E)$, d_A is the differential associated to A on $\mathrm{Ad}E$ and G_A Green's operator of A on $\Omega^0(\mathbb{R}^4; \mathrm{Ad}E)$.

For the quotient connection ∇^{∞}, the curvature is given as follows.

Proposition 2.3.3 *The three components of the curvature of ∇^{∞} in $\mathrm{Ad}\mathbb{P} \longrightarrow \tilde{\mathcal{B}}(k) \times \mathbb{R}^4$ are given at the point $([A], x)$ by*

1. $F(\nabla^{\infty})(X, Y) = F(A)(X, Y)$,
2. $F(\nabla^{\infty})((a, s_e), X) = \Pi(X \lrcorner a)$,
3. $F(\nabla^{\infty})((a, s_e), (a', s_{e'})) = -2\Pi G_A\{a, a'\}(x)$,

where $X, Y \in \mathrm{T}_x\mathrm{S}^4$, $a, a' \in \Omega^1(\mathrm{S}^4; \mathrm{Ad}E) \cap \ker d_A^*$, $e, e' \in \mathrm{Ad}E_{\infty}$ and $\{\cdot, \cdot\}$ is the pairing derived from the metric on 1-forms and the Lie bracket on endomorphisms.

Now, the derivative of the Atiyah map

$$\mathrm{dAt}_A : \mathrm{T}_{(T,P)}\mathbf{A}^*(k) \longrightarrow \mathrm{T}_{A(T,P)}\mathcal{A}(k) = \Omega^1(\mathrm{S}^4; \mathrm{Ad}E)$$

does not respect the gauge fixing conditions. That is, the subspace of $\mathrm{T}_{(T,P)}\mathbf{A}^*(k)$ solving (2.12) does not map into the subspace of $\Omega^1(\mathrm{S}^4; \mathrm{Ad}E)$ solving $\mathrm{d}_A^* a = 0$. However Corrigan, Goddard and Templeton [11] and Osborne [26], showed that a correction term can be applied so that the image of the horizontal subspace of $\mathrm{T}_{(T,P)}\mathbf{A}^*(k)$ under dAt_A could be mapped isomorphically via a projection onto $\ker \mathrm{d}_A^*$. For the reduced moduli space, their results show that the tangent vector associated to (t, p) in the horizontal subspace of $\mathrm{T}_{(T,P)}\mathbf{A}^*(k)$ is given by

$$Z(t, p) = \sigma^{-2}\left(P^*\tilde{y}^*\mathrm{d}x\mathfrak{I}p - p^*\mathfrak{I}\mathrm{d}x^*\tilde{y}P + P^*\tilde{y}^*\left(\mathrm{d}x\mathfrak{I}t^* - t\mathfrak{I}\mathrm{d}x^*\right)\tilde{y}P\right). \tag{2.17}$$

Hence for the framed space a tangent vector will be given by

$$\widetilde{Z} = (Z, Z_{\infty}).$$

2.3.5 Comparing Curvatures

Recall that at the end of §2.3.3 that we derived the following formula for the curvature of ∇

$$\begin{aligned} F(\nabla) &= v^*\begin{bmatrix} \mathrm{d}\tilde{x}\wedge\mathfrak{I}\mathrm{d}\tilde{x}^* & -\mathrm{d}\tilde{x}\wedge\mathfrak{I}\mathrm{d}P \\ -\mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}\tilde{x}^* & \mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}P\end{bmatrix}v\Big|_H \\ &+ v^*\begin{bmatrix} -\Phi^{-1}\left(\widehat{\mathrm{d}\tilde{\theta}^1}\right) & 0 \\ 0 & 0\end{bmatrix}v\Big|_H. \end{aligned}$$

We will now try to compare this with the curvature $F(\nabla^\infty)$ using Proposition 2.3.3. In order to do this, we split $F(\nabla)$ into its components with respect to the splitting

$$\begin{aligned} \Lambda^1(M_k\times\mathbb{R}^4)\otimes\Lambda^1(M_k\times\mathbb{R}^4) &= \Lambda^1(M_k)\otimes\Lambda^1(M_k) \\ &\oplus \Lambda^1(M_k)\otimes\Lambda^1(\mathbb{R}^4) \\ &\oplus \Lambda^1(\mathbb{R}^4)\otimes\Lambda^1(M_k) \\ &\oplus \Lambda^1(\mathbb{R}^4)\otimes\Lambda^1(\mathbb{R}^4), \end{aligned}$$

to get

$$\begin{aligned} F(\nabla) &= v^*\begin{bmatrix} \mathrm{d}x\wedge\mathfrak{I}\mathrm{d}x^* & 0 \\ 0 & 0\end{bmatrix}v\Big|_H \\ &+ v^*\begin{bmatrix} -\mathrm{d}T\wedge\mathfrak{I}\mathrm{d}x^*-\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}T^* & -\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}P \\ -\mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}x^* & 0\end{bmatrix}v\Big|_H \\ &+ v^*\begin{bmatrix} \mathrm{d}T\wedge\mathfrak{I}\mathrm{d}T^*-\Phi^{-1}\widehat{\mathrm{d}\tilde{\theta}^1} & \mathrm{d}T\wedge\mathfrak{I}\mathrm{d}P \\ \mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}T^* & \mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}P\end{bmatrix}v\Big|_H. \end{aligned}$$

It is obvious that the first term in the sum is indeed the curvature of the connection $A(T,P)$. The second term is also very interesting. Expanding

$$\begin{aligned} & v^*\begin{bmatrix} -\mathrm{d}T\wedge\mathfrak{I}\mathrm{d}x^*-\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}T^* & -\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}P \\ -\mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}x^* & 0\end{bmatrix}v \\ =\;& -\sigma^{-2}\left(P^*\tilde{y}^*\left(\mathrm{d}T\wedge\mathfrak{I}\mathrm{d}x^*+\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}T^*\right)\tilde{y}P+P^*\tilde{y}^*\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}P-\mathrm{d}P^*\wedge\mathfrak{I}\mathrm{d}x^*\tilde{y}P\right) \end{aligned}$$

where $\tilde{y}=\tilde{x}^{*-1}$. Now we are interested primarily in examining

$$F(\nabla)((0,t,p),(X,0,0))$$

for $X\in\Omega^0(\mathbb{R}^4;\mathrm{T}\mathbb{R}^4)$ and $(t,p)\in\ker\theta$. Observe that

$$\begin{aligned} (\mathrm{d}T\wedge\mathfrak{I}\mathrm{d}x^*)\left((0,t,p),(X,0,0)\right) &= (\mathrm{d}T_i\wedge\mathfrak{I}\mathrm{d}x_i)\left((0,t,p),(X,0,0)\right)q_iq_j^* \\ &= (t_i\mathfrak{I}X_j)\,q_iq_j^* \;=\; t\mathfrak{I}X^*. \end{aligned}$$

Similarly

$$(\mathrm{d}x\wedge\mathfrak{I}\mathrm{d}T^*)\left((0,t,p),(X,0,0)\right) = -X\mathfrak{I}t^*,$$

etc.

Hence

$$\begin{aligned}
& F(\nabla)\left((0,t,p),(X,0,0)\right) \\
= & -\sigma^{-2}\left(P^*\tilde{y}^*(t\mathfrak{F}X^* - X\mathfrak{F}t^*)\tilde{y}P - P^*\tilde{y}X\mathfrak{F}p + p^*\mathfrak{F}X^*\tilde{y}P\right) \\
= & -\sigma^{-2}\left(P^*\tilde{y}^*(t\mathfrak{F}\mathrm{d}x^* - \mathrm{d}x\mathfrak{F}t^*)\tilde{y}P - P^*\tilde{y}\mathrm{d}x\mathfrak{F}p + p^*\mathfrak{F}\mathrm{d}x^*\tilde{y}P\right)(X).
\end{aligned}$$

We find that

$$F(\nabla)\left((0,t,p),(X,0,0)\right) = X \lrcorner Z(t,p).$$

Our next painful task is to take the laplacian of $F(\widehat{\nabla})$ restricting to the T and P directions and then compare this with the $\{,\}$ product of tangent vectors to the Moduli space.

Now, for a general $M \in \Omega^0(\mathbb{R}^4; \underline{\mathrm{End}(\mathbb{R}^k) \otimes \mathbb{H}})$, we have

$$\begin{aligned}
\Delta_A v^* M v &= -v^*\left(\partial_{ii} M - 4\iota\mathfrak{F}\iota^* M - 4M\iota\mathfrak{F}\iota^*\right. \\
&+ 2\iota\mathfrak{F}q_i\mathfrak{R}\partial_i M + 2\partial_i M\mathfrak{R}^* q_i{}^*\mathfrak{F}\iota^* \\
&+ \left. 2\iota\mathfrak{F}q_i\mathfrak{R}M\mathfrak{R}^* q_i{}^*\mathfrak{F}\iota\right) v,
\end{aligned} \tag{2.18}$$

and if M is constant with respect to the x_is,

$$\Delta_A v^* M v = v^*\left(-4\iota\mathfrak{F}\iota^* M - 4M\iota\mathfrak{F}\iota^* + 2\iota\mathfrak{F}q_i\mathfrak{R}M\mathfrak{R}^* q_i{}^*\mathfrak{F}\iota\right) v. \tag{2.19}$$

Applying this to

$$M = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix},$$

we find

$$\begin{aligned}
\Delta_A v^* \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} v &= -v^* \begin{bmatrix} 8\mathfrak{F}\tilde{x}_i K\tilde{x}_i\mathfrak{F} - 4\mathfrak{F}K - 4K\mathfrak{F} & 0 \\ 0 & 0 \end{bmatrix} v \\
&= -v^* \begin{bmatrix} 4\mathfrak{F}\Phi(K)\mathfrak{F} & 0 \\ 0 & 0 \end{bmatrix} v,
\end{aligned}$$

i.e.

$$\Delta_A M = -4v^*\iota\mathfrak{F}\Phi(K)\mathfrak{F}\iota^* v. \tag{2.20}$$

Similarly for

$$M = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix},$$

we have

$$\Delta_A v^* M v = -8v^*\iota\mathfrak{F}\Re(PLP^*)\mathfrak{F}\iota^* v. \tag{2.21}$$

This accounts for the curvature of the bundle $\mathbf{A}^*(k) \longrightarrow \tilde{M}_k$.

Now, we wish to calculate

$$\{Z,Z\} + \frac{1}{2}\Delta_A v^* \left(\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R} - M\right) v,$$

where $M = \Psi_1 \widehat{\mathrm{d}}\theta^1$. To try and simplify this calculation let us adopt the notation

$$\begin{aligned} \mathfrak{i}(B) &= B - B^{(*)}, \\ \mathfrak{r}(B) &= B + B^{(*)}, \end{aligned}$$

for matrices B with entries in $\Lambda^\bullet(\mathrm{T}^*\mathbb{R}^4 \times \mathbf{A}^*(k)) \otimes \mathbb{H}$, where

$$(B \wedge B')^{(*)} = B'^{(*)} \wedge B^{(*)},$$

and

$$B^{(*)} = B^*$$

if B is a matrix of 0-forms.

Then we have

$$\begin{aligned} \{Z,Z\} &= (v^*\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*v - v^*\iota q_i\mathcal{F}\delta\mathcal{R}v) \wedge (v^*\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*v - v^*\iota q_i\mathcal{F}\delta\mathcal{R}v) \\ &= v^*\Big(\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*) - \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* - 4\delta\mathcal{R}^*\mathcal{F}\mathcal{F}\delta\mathcal{R} \\ &- \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*\mathcal{R}^*\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*) + \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* \\ &+ \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*\mathcal{R}^*\mathcal{F}\mathcal{R}\iota q_i\mathcal{F} \wedge \delta\mathcal{R}\Big)v. \end{aligned}$$

So

$$\begin{aligned} \{Z,Z\} &= v^*\Big(\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}q_i{}^* \wedge \mathrm{d}T\mathcal{F}q_i{}^*\iota^*) - \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* + \frac{1}{2}\delta\mathcal{R}^*(\partial_{ii}\mathcal{F})\delta\mathcal{R} \\ &- 4\delta\mathcal{R}^*\mathcal{F}\tilde{x}_i\mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R} + \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}q_i{}^*\tilde{x}\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*) \\ &+ \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* + \delta\mathcal{R}^*\mathcal{F}q_i{}^*\tilde{x}F\tilde{x}^*\mathcal{F}q_i \wedge \delta\mathcal{R}\Big)v \\ &= v^*\Big(- 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F} \wedge \mathrm{d}T^*\mathcal{F}\iota^*) - \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* \\ &+ \frac{1}{2}\delta\mathcal{R}^*(\partial_{ii}\mathcal{F})\delta\mathcal{R} - 4\delta\mathcal{R}^*\mathcal{F}\tilde{x}_i\mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R} + \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}q_i{}^*\tilde{x}\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*) \\ &+ \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* + 4\delta\mathcal{R}^*\mathcal{F}\tilde{x}_i\mathcal{F}\tilde{x}_i\mathcal{F} \wedge \delta\mathcal{R}\Big)v, \end{aligned}$$

and hence

$$\begin{aligned} \{Z,Z\} &= v^*\Big(\frac{1}{2}\delta\mathcal{R}^* \wedge (\partial_{ii}\mathcal{F})\delta\mathcal{R} - 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F} \wedge \mathrm{d}T^*\mathcal{F}\iota^*) - \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* \\ &+ \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^* + \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}q_i{}^*\tilde{x}\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*)\Big)v. \end{aligned}$$

We also find that

$$\begin{aligned}
-\Delta_A v^* \delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R} v &= v^*\Big(\delta\mathcal{R}^* \wedge (\partial_{ii}\mathcal{F})\delta\mathcal{R} - 4\mathfrak{t}(\iota\mathcal{F}\iota^*\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}) \\
&+ 2\mathfrak{t}(\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge (\partial_i\mathcal{F})\delta\mathcal{R}) + 2\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}\mathcal{R}^*{q_i}^*\mathcal{F}\iota\Big)v \\
&= v^*\Big(\delta\mathcal{R}^* \wedge (\partial_{ii}\mathcal{F})\delta\mathcal{R} - 4\mathfrak{t}(\iota\mathcal{F}\mathrm{d}T \wedge \mathcal{F}\delta\mathcal{R}) \\
&- 4\mathfrak{t}(\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R}) + 2\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}\mathcal{R}^*{q_i}^*\mathcal{F}\iota\Big)v.
\end{aligned}$$

Hence

$$\begin{aligned}
&\{Z,Z\} + \frac{1}{2}\Delta_A v^*\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}v \\
=\ & v^*\Big(\frac{1}{2}\delta\mathcal{R}^*(\partial_{ii}\mathcal{F})\delta\mathcal{R} - 2\mathfrak{t}(\delta\mathcal{R}^*\mathcal{F} \wedge \mathrm{d}T^*\mathcal{F}\iota^*) \\
-\ & \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* + \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* \\
+\ & \mathfrak{t}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F}\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^*) \\
-\ & \frac{1}{2}\delta\mathcal{R}^* \wedge (\partial_{ii}\mathcal{F})\delta\mathcal{R} + 2\mathfrak{t}(\iota\mathcal{F}\mathrm{d}T \wedge \mathcal{F}\delta\mathcal{R}) \\
+\ & 2\mathfrak{t}(\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R}) \\
-\ & \iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}\mathcal{R}^*{q_i}^*\mathcal{F}\iota\Big)v.
\end{aligned}$$

This gives

$$\begin{aligned}
&\{Z,Z\} + \frac{1}{2}\Delta_A v^*\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}v \\
=\ & v^*\Big(- \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* + \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F} \wedge \mathcal{R}\delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* \\
+\ & \mathfrak{t}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F} \wedge \mathcal{R}\delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^*) \\
+\ & 2\mathfrak{t}(\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R}) - \iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}\mathcal{R}^*{q_i}^*\mathcal{F}\iota\Big)v.
\end{aligned}$$

Now apply the fact that

$$\mathcal{R}\delta\mathcal{R}^* = \delta(\mathcal{R}\mathcal{R}^*) - \delta\mathcal{R}\mathcal{R}^*$$

to see

$$\begin{aligned}
&\{Z,Z\} + \frac{1}{2}\Delta_A v^*\delta\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}v \\
=\ & v^*\Big(- \iota q_i\mathcal{F}\delta\mathcal{R} \wedge \delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* \\
+\ & \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F} \wedge \delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}{q_i}^*\iota^* - \iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F} \wedge \delta\mathcal{R}\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* \\
+\ & \mathfrak{t}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F} \wedge \delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}{q_i}^*\iota^*) - \mathfrak{t}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F} \wedge \delta\mathcal{R}\mathcal{R}^*\mathcal{F}{q_i}^*\iota^*) \\
+\ & 2\mathfrak{t}(\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^* \wedge \mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R}) \\
-\ & \iota\mathcal{F}q_i\delta(\mathcal{R}\mathcal{R}^*) \wedge \mathcal{F}\delta\mathcal{R}\mathcal{R}^*{q_i}^*\mathcal{F}\iota + \iota\mathcal{F}q_i\delta\mathcal{R}\mathcal{R}^* \wedge \mathcal{F}\delta\mathcal{R}\mathcal{R}^*{q_i}^*\mathcal{F}\iota\Big)v.
\end{aligned}$$

Hence

$$\begin{aligned}
\{Z,Z\}+\frac{1}{2}\Delta_A v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v &= v^*\Big(-\iota q_i\mathcal{F}\delta\mathcal{R}\wedge\delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* \\
&+ \mathfrak{i}(\iota q_i\mathcal{F}\delta\mathcal{R}\mathcal{R}^*\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}{q_i}^*\iota^*) \\
&+ \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}{q_i}^*\iota^*) \\
&- \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F}\wedge\delta\mathcal{R}\mathcal{R}^*\mathcal{F}{q_i}^*\iota^*) \\
&+ 2\mathfrak{r}(\iota\mathcal{F}q_i\mathcal{R}\delta\mathcal{R}^*\wedge\mathcal{F}\tilde{x}_i\mathcal{F}\delta\mathcal{R})\Big)v.
\end{aligned}$$

Here, we have freely used the fact that $\mathcal{F}$ commutes with the quaternions. We also note for further use that $\delta(\mathcal{R}\mathcal{R}^*)$ is also a real $k\times k$ matrix, given by the ADHM condition. Now, recall that the condition that the pair $t\in\odot^2(\mathbb{R}^k)\otimes\mathbb{H}, p\in\mathbb{H}^k$ were horizontal tangent vectors is given by (2.14):

$$\mathcal{R}\begin{bmatrix} t \\ p^* \end{bmatrix}\in\odot^2(\mathbb{R}^k)\otimes\mathbb{H}.$$

This is better expressed as

$$\mathcal{R}\delta\mathcal{R}^*\in\odot^2(\mathbb{R}^k)\otimes\mathbb{H}$$

and, still further, as

$$\delta(\mathcal{R}\mathcal{R}^*)=2\Re(\delta\mathcal{R}\mathcal{R}^*) \tag{2.22}$$

We also use the identities for quaternions r, s

$$q_i r^* q_i = -2r \tag{2.23}$$

and

$$q_i r^* s {q_i}^* = 4\Re(r^* s). \tag{2.24}$$

In particular

$$q_i\delta\mathcal{R}\mathcal{R}^*{q_i}^* = 4\Re(\delta\mathcal{R}\mathcal{R}^*) = 2\delta(\mathcal{R}\mathcal{R}^*) \tag{2.25}$$

by (2.24). Hence

$$\begin{aligned}
\{Z,Z\}+\frac{1}{2}\Delta_A v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v &= v^*\Big(-\iota q_i\mathcal{F}\delta\mathcal{R}\wedge\delta\mathcal{R}^*\mathcal{F}{q_i}^*\iota^* \\
&+ 2\mathfrak{i}(\iota\mathcal{F}\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}^*\iota^*) && \text{by (2.25) and (2.22)} \\
&- 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}^*\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}^*\iota^*) && \text{since } \mathcal{R}\mathcal{R}^* \text{ is real} \\
&- \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}{q_i}^*\tilde{x}\mathcal{F}\wedge\delta\mathcal{R}\mathcal{R}^*\mathcal{F}{q_i}^*\iota^*) \\
&+ 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}_i\mathcal{F}\wedge\delta\mathcal{R}\mathcal{R}^*q_i^*\mathcal{F}\iota)\Big).
\end{aligned}$$

By the definition of i

$$\begin{aligned}\{Z,Z\}+\frac{1}{2}\Delta_A v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v &= v^*\Big(-\iota q_i\mathcal{F}\delta\mathcal{R}\wedge\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*\\ &+ 0\\ &- 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}^*\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}^*\iota^*)\\ &+ \mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}^*q_i\mathcal{F}\wedge\delta\mathcal{R}\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*)\\ &- 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}_i\mathcal{F}\wedge\delta\mathcal{R}\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*)\\ &\qquad\text{using properties of Clifford multiplication}\\ &+ 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}_i\mathcal{F}\wedge\delta\mathcal{R}\mathcal{R}^*q_i^*\mathcal{F}\iota)\Big)v\end{aligned}$$

and thus finally we see that

$$\begin{aligned}\{Z,Z\}+\frac{1}{2}\Delta_A v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v &= v^*\Big(-\iota q_i\mathcal{F}\delta\mathcal{R}\wedge\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*\\ &- 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}^*\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R}^*)\mathcal{F}^*\iota^*)\\ &+ 2\mathfrak{r}(\delta\mathcal{R}^*\mathcal{F}\tilde{x}^*\mathcal{F}\wedge\delta(\mathcal{R}\mathcal{R})^*\mathcal{F}^*\iota^*)\Big)v\\ &= v^*\Big(-\iota q_i\mathcal{F}\delta\mathcal{R}\wedge\delta\mathcal{R}^*\mathcal{F}q_i{}^*\iota^*\Big)v\\ &\qquad\text{by (2.25) and (2.22)}\\ &= -\frac{1}{2}\Delta_A v^*\iota\Phi^{-1}(\widehat{\mathrm{d}}\tilde{\theta}^1)\iota^*v.\end{aligned}$$

So we have proved that

$$\Delta_A\left(v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v-v^*\iota\Phi^{-1}(\widehat{\mathrm{d}}\tilde{\theta}^1)\iota^*v\right)=-2\{Z,Z\}. \tag{2.26}$$

That is,

$$v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v-v^*\iota\Phi^{-1}(\widehat{\mathrm{d}}\tilde{\theta}^1)\iota^*v=-2G_A\{Z,Z\}+\text{a harmonic section of }\mathrm{Ad}E \tag{2.27}$$

or, to put it another way,

$$v^*\delta\mathcal{R}^*\wedge\mathcal{F}\delta\mathcal{R}v-v^*\iota\Phi^{-1}(\widehat{\mathrm{d}}\tilde{\theta}^1)\iota^*v=-2\Pi G_A\{Z,Z\} \tag{2.28}$$

since the left hand side evaluates to 0 at infinity.

Hence we have the following result.

Theorem 2.3.4 *1.* $F(\nabla)\left((X,0,0),(Y,0,0)\right)=F(A(T,P))_x(X,Y)$;

2. $F(\nabla)\left((X,0,0),(0,t,p)\right)=\Pi\left(X\lrcorner Z(t,p)\right)$;

3. $F(\nabla)\left((0,t_1,p_1),(0,t_2,p_2)\right)=-2\Pi G_A\{Z(t_1,p_1),Z(t_2,p_2)\}$.

We have found a connection on $\widehat{E}$ so that the same universal construction yields the same curvatures for the space of ASD connections and the space of ADHM data.

Remark 2.3.5 *Examining the diffeomorphism between* $\widetilde{M}_k$ *and* $\widetilde{\mathcal{M}}_k$ *we see that for* $\alpha \in \mathrm{Sp}(1)$

$$[T, P\alpha^{-1}] \mapsto \mathrm{Ad}(\alpha)A([T,P])$$

implying that the diffeomorphism is $\mathrm{Sp}(1)$ *invariant. This gives us the added bonus that if we correct for the* $\mathrm{Sp}(1)$ *action as well, we end up with the same representatives of* $c_2(\mathbb{E})$ *on the reduced space* $\mathcal{M}_k$.

Hopefully the ADHM construction will now yield more calculable results. To reduce the case to the unframed moduli space, we will use equivariant cohomology on our previous work.

2.3.6 Reasons for the Anomalies

Let us examine what is going on more closely. The bundle $\widehat{E}$ is defined on $\mathbb{R}^4 \times \mathfrak{M}_{\mathbb{C}}^k$ minus the set S_k consisting of the points $(x, T, P) \in \mathbb{R}^4 \times \mathfrak{M}_{\mathbb{C}}^k$ such that there is $u \in \mathrm{U}(k)$ for which

$$uTu^{-1} = \begin{pmatrix} T' & 0 \\ 0 & x \end{pmatrix}, \quad uP = \begin{pmatrix} P' \\ 0 \end{pmatrix},$$

where $(T', P') \in \mathfrak{M}_{\mathbb{C}}^{k-1}$. We are then integrating a form on the quotient of the hyperKähler reduction of $\left(\mathbb{R}^4 \times \mathfrak{M}_{\mathbb{C}}^k\right) \backslash S_k$, i.e.

$$\mathbb{M}_k = \frac{\left((\mathbb{R}^4 \times \mathfrak{M}_{\mathbb{C}}^k)\backslash S_k\right) /\!\!/\!\!/ \mathrm{U}(k)}{\mathrm{SO}(3)}.$$

(We will deal with hyperKähler geometry in the next chapter.)

In the infinite dimensional case, for a submanifold Σ of $\mathbb{R}^4$ we form

$$\mu_{Don}(\Sigma) = c_2(\mathbb{E})/[\Sigma] = \int_\Sigma c_2(\mathbb{E}) = \int_{\mathbb{R}^4} c_2(\mathbb{E}) \wedge \pi^* \mathrm{PD}(\Sigma_i) \in \mathrm{H}^{4-\dim\Sigma}(\mathcal{M}^*)$$

where $c_2(\mathbb{E}) \in \mathrm{H}^4(\mathbb{R}^4 \times \mathcal{M}^*)$ and $\pi : \mathbb{R}^4 \times \mathcal{M} \longrightarrow \mathbb{R}^4$ is the projection. The slant product actually works because we have a perfectly decent trivial fibration $\mathbb{R}^4 \times \mathcal{M}^* \longrightarrow \mathbb{R}^4$.

However in the finite case the fibration

$$\mathbb{M}_k \longrightarrow \mathbb{R}^4$$

is destroyed because we remove points (namely S_k) from the direct product before taking the various quotients. If we naïvely form the finite dimensional version

$$\mu_{Don}(\Sigma) = \frac{-1}{8\pi^2} \int_\Sigma \mathrm{tr}(F(\nabla) \wedge F(\nabla))$$

then it is not altogether clear where this $\mu_{Don}(\Sigma)$ lies. The Chern-Weil representative of $\mu_{Don}(\Sigma)$ doesn't really represent a cohomology class on $\mathcal{M}$. We are forced therefore to reinterpret the situation for the ADHM case.

We have the 4-form c representing the second Chern class of the universal bundle $\widehat{E}$ over the manifold $\mathbb{M}_k$. Let $\Sigma_1, \ldots, \Sigma_k$ be compact submanifolds of $\mathbb{R}^4$ without boundary whose dimensions sum to $4l - 8k + 3$, that is of the correct dimensions to form a Donaldson polynomial. We can then consider the form

$$\mu_i = c \wedge \iota^* \pi^* \mathrm{PD}(\Sigma_i)$$

where $\pi : \mathbb{R}^4 \times \mathcal{M}_k \longrightarrow \mathbb{R}^4$ is projection, and $\iota : \mathbb{M}_k \hookrightarrow \mathbb{R}^4 \times \mathcal{M}_k$ inclusion. Thus we can form the integral

$$\int_{D_{(k,l)}} \Delta^* \left(\Pi_1^* \mu_1 \wedge \ldots \wedge \Pi_l^* \mu_l \right)$$

where $\Pi_i : \mathbb{M}_k^l \longrightarrow \mathbb{M}_k$ is projection onto the ith factor,

$$D_{(k,l)} = \{(x_1, \ldots, x_l, [T, P]) \in (\mathbb{R}^4)^l \times \mathcal{M}_k | (x_i, [T, P]) \in \mathbb{M}_k \text{ for all } k\}$$

and

$$\begin{gathered} \Delta : D_{(k,l)} \hookrightarrow (\mathbb{R}^4 \times \mathcal{M}_k)^l \\ (x_1, \ldots, x_l, [T, P]) \mapsto ((x_1, [T, P]), (x_2, [T, P]), \ldots, (x_l, [T, P])) \end{gathered}$$

a sort of diagonal map.

A similar method with the infinite dimensional case yields the construction of the Donaldson polynomial. We will go on and show that although this integral does not agree for the $k = 1$ case where there is a discrepancy manifesting in the linking phenomena, there is no further discrepancy for higher k.

Chapter 3

Equivariant Cohomology and Quotients of Symplectic and HyperKähler Manifolds

We have tried to work out the Donaldson μ-map via the ADHM construction. However, we have found that the matrices that occur in the construction contain enormous polynomials which give no insight as to what form the μ-map should take. We use equivariant cohomology introduced by [5] and the localisation developed by [14] and [18] applied to hyperKähler reductions introduced by [17] to try and obtain some answers to the integrals involved.

3.1 A Review of Equivariant Cohomology

3.1.1 The General Theory

Let G be a compact connected Lie Group acting on a manifold M. Associated to G is the classifying space BG and the universal bundle $EG \longrightarrow BG$. We define the equivariant cohomology of the space M by

$$H_G^\bullet(M) = H^\bullet(EG \times_G M).$$

From this we see that the G-equivariant cohomology of the point

$$H_G^\bullet(\mathrm{pt}) = H^\bullet(EG \times_G \mathrm{pt}) = H^\bullet(BG) = H_G^\bullet$$

is very large. Also, if the the action of G is free on M then because EG is contractible

$$\pi : EG \times_G M \longrightarrow M/G$$

induces an isomorphism

$$\pi^* : H^\bullet(M/G) = H^\bullet(M/G \times EG) \xrightarrow{\cong} H^\bullet(EG \times_G M) = H_G^\bullet(M). \tag{3.1}$$

We remark that for a compact connected Lie Group G with maximal torus T, we have the relation

$$H_G^\bullet(M) \cong H_T^\bullet(M)^W \tag{3.2}$$

where W is the Weyl group of T in G. It therefore makes sense to pass from the full group G to the maximal torus T. Any integrals can be amended using the Weyl integration formula, see [10, 18].

Theorem 3.1.1 *[Weyl Integration Formula] For a compact connected Lie group G with maximal torus T and any G-invariant function $f : G \longrightarrow \mathbb{R}$, we have*

$$\int_{\mathfrak{g}} f(\phi)[\mathrm{d}\phi] = \frac{\operatorname{vol} G}{|W| \operatorname{vol} T} \int_{\mathfrak{t}} f(\psi) w(\psi)^2 [\mathrm{d}\psi],$$

where W is the Weyl group, and w the product of the positive roots.

We can calculate this cohomology using the de Rham model where the complex is defined by

$$\ldots \longrightarrow \Omega_G^{p-1}(M) \xrightarrow{\mathrm{d}_{\mathfrak{g}}} \Omega_G^p(M) \xrightarrow{\mathrm{d}_{\mathfrak{g}}} \Omega_G^{p+1}(M) \longrightarrow \ldots$$

The spaces are

$$\Omega_G^\bullet(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M))^G$$

where $\mathbb{C}[\mathfrak{g}]$ denotes the symmetric polynomials in the Lie algebra $\mathfrak{g}$ and the differential $\mathrm{d}_{\mathfrak{g}}$ is given by

$$(\mathrm{d}_{\mathfrak{g}}\alpha)(\xi) = \mathrm{d}(\alpha(\xi)) - X_\xi \lrcorner (\alpha(\xi))$$

for all $\xi \in \mathfrak{g}$, where X_ξ is the vector field on M generated by ξ. It is easily checked that $\mathrm{d}_{\mathfrak{g}}$ squares to zero precisely on $\Omega_G^\bullet(M)$ so that we have a complex. The (co)homology of this complex gives rise to the de Rham model of the equivariant cohomology of M.

3.1.2 Using the de Rham Model

We can see the isomorphism (3.1) in the de Rham model in the case of a free G-action as follows. We have the principal G-bundle $p : M \longrightarrow M/G$, hence a mapping of forms $p^* : \Omega^\bullet(M/G) \longrightarrow \Omega^\bullet(M)$.

Given a form $\eta \in \Omega^\bullet(M/G)$, we know that $p^*\eta$ satisfies the following conditions:

1. $p^*\eta$ is G-invariant,

2. $X_\xi \lrcorner p^*\eta = 0$.

Hence if η is closed, then $p^*\eta$ is equivariantly closed. We therefore have a map of cohomologies

$$p^* : H^\bullet(M/G) \longrightarrow H^\bullet_G(M).$$

We know that p^* is injective since p is a principal fibration, so we would like to show that it is surjective, that is:

Theorem 3.1.2 *[[14, 8]] For an equivariantly closed form $\alpha \in \Omega^k_G(M)$ there are forms $\beta \in \Omega^{k-1}_G(M)$, $\gamma \in \Omega^k(M/G)$, such that*

$$\alpha = \mathrm{d}_{\mathfrak{g}}\beta + p^*\gamma. \tag{3.3}$$

Proof

We prove this for the case $G = T$ the s-dimensional torus. For a general compact, connected G, we can just apply the isomorphism (3.2). Let $\xi_1 \dots \xi_s$ be an orthonormal basis for $\mathfrak{g}^*$, with dual basis $e_1 \dots e_s$. Since

$$\mathbb{C}[\mathfrak{g}] = \bigodot_{j=1}^{s} \mathbb{C}[\xi_j],$$

we can restrict ourselves to the case $G = S^1$ by considering the torus generated by a single e_i. The proof then follows exactly the argument in Proposition 2.1 of [14]. Essentially, a free action of G means that the induced vector fields X_ξ do not vanish. Hence we can, in some sense, "invert" certain equivariant forms on M away from the fixed set M_0, which contain the equivariant form $v \mapsto |X_v|^2$ when we put a Riemannian metric on M such that $|X_{e_i}| = |e_i| = 1$. Indeed, we can put a G-invariant Riemannian metric on M with this property and set

$$\theta(v) = \langle X_v, \cdot \rangle = \xi_i(v) \langle X_{e_i}, \cdot \rangle . \tag{3.4}$$

Then note that $\mathrm{d}_{\mathfrak{g}}\theta = \mathrm{d}\theta - \xi_i$ has formal inverse

$$\begin{aligned} (\mathrm{d}_{\mathfrak{g}}\theta)^{-1} &= -\xi_i^{-1}(1 - \xi_i^{-1}\mathrm{d}\theta)^{-1} \\ &= -\xi_i^{-1} \sum_{j=0}^{\infty} \xi_i^{-j} \mathrm{d}\theta^j . \end{aligned}$$

We also notice that since θ is effectively a connection, $\mathrm{d}\theta$ is effectively the curvature which is a horizontal form. This could also be proved by expanding $0 = {\mathrm{d}_{\mathfrak{g}}}^2\theta$.

Now let α be an equivariantly closed k-form on M, then set

$$\tilde{\beta} = \frac{\alpha\theta}{\mathrm{d}_{\mathfrak{g}}\theta}.$$

Let $\alpha = \sum_r \alpha_r \xi_i^r$ with $\deg \alpha_r = k - 2r$. Since α is equivariantly closed, the α_r satisfy

$$\mathrm{d}\alpha_0 + \sum_{r\geq 1} \left(\mathrm{d}\alpha_r - e_i \lrcorner \alpha_{r-1}\right) \xi_i^r = 0,$$

that is

$$
\begin{aligned}
\mathrm{d}\alpha_0 &= 0 \\
\mathrm{d}\alpha_r &= e_i \lrcorner \alpha_{r-1}
\end{aligned} \tag{3.5}
$$

for all r.

We have

$$
\tilde{\beta} = -\sum_{r\geq 0}\sum_{j=0}^{\infty} \alpha_r \theta \xi_i^{(r-j-1)} \mathrm{d}\theta^j.
$$

Now, not all the powers of ξ_i^2 are positive. Let

$$
\beta = \sum_{r\geq 1}\sum_{j=0}^{r-1} \alpha_r \theta \xi_i^{(r-j-1)} \mathrm{d}\theta^j,
$$

and

$$
\tilde{\gamma} = -\sum_{r\geq 0} \alpha_r \theta \mathrm{d}\theta^r.
$$

Now, by (3.5),

$$
e_i \lrcorner \tilde{\gamma} = -\sum_{r\geq 1} \mathrm{d}\alpha_r \theta \mathrm{d}\theta^{r-1} - (-1)^k \sum_{r\geq 0} \alpha_r \mathrm{d}\theta^r,
$$

and

$$
\begin{aligned}
\mathrm{d}_{\mathfrak{g}}\beta &= \sum_{r\geq 1}\sum_{j=0}^{r-1} \mathrm{d}\alpha_r \theta \xi_i^{(r-j-1)} \mathrm{d}\theta^j - \sum_{r\geq 1}\sum_{j=0}^{r-1} \mathrm{d}\alpha_{r+1} \theta \xi_i^{(r-j)} \mathrm{d}\theta^j \\
&+ (-1)^k \sum_{r\geq 1}\sum_{j=0}^{r-1} \alpha_r \xi_i^{(r-j-1)} \mathrm{d}\theta^{j+1} - (-1)^k \sum_{r\geq 1}\sum_{j=0}^{r-1} \alpha_r \xi_i^{(r-j)} \mathrm{d}\theta^j \\
&= \sum_{r\geq 1}\sum_{j=0}^{r-1} \mathrm{d}\alpha_r \theta \xi_i^{(r-j-1)} \mathrm{d}\theta^j - \sum_{r\geq 2}\sum_{j=0}^{r-2} \mathrm{d}\alpha_r \theta \xi_i^{(r-j-1)} \mathrm{d}\theta^j \\
&+ (-1)^k \sum_{r\geq 1}\sum_{j=1}^{r} \alpha_r \xi_i^{(r-j)} \mathrm{d}\theta^j - (-1)^k \sum_{r\geq 1}\sum_{j=0}^{r-1} \alpha_r \xi_i^{(r-j)} \mathrm{d}\theta^j \\
&= \sum_{r\geq 1} \mathrm{d}\alpha_r \theta \mathrm{d}\theta^{r-1} + (-1)^k \sum_{r\geq 1} \alpha_r \mathrm{d}\theta^r - (-1)^k \sum_{r\geq 1} \alpha_r \xi_i^r \\
&= \sum_{r\geq 1} \mathrm{d}\alpha_r \theta \mathrm{d}\theta^{r-1} + (-1)^k \sum_{r\geq 0} \alpha_r \mathrm{d}\theta^r - (-1)^k \sum_{r\geq 0} \alpha_r \xi_i^r \\
&= -e_i \lrcorner \tilde{\gamma} + (-1)^k \alpha.
\end{aligned}
$$

Since $e_i \lrcorner \tilde{\gamma}$ is equivariant and $e_i \lrcorner \left(e_i \lrcorner \tilde{\gamma}\right) = 0$, it must be the pull back of a form on M/G under p, i.e.

$$
e_i \lrcorner \tilde{\gamma} = p^* \gamma
$$

for some $\gamma \in \Omega^k(M/G)$.

Hence

$$\alpha = (-1)^k \mathrm{d}_{\mathfrak{g}}\beta + (-1)^k p^*\gamma$$

as required. ■

This gives the isomorphism of cohomologies explicitly in the de Rham model. We will use this when we come to equivariant integration.

3.1.3 Subspaces with Non-trivial Stabiliser

Let M be a manifold with an action of the compact Lie group G. Very frequently, the action of G is not free on M and so we need some way of looking at the subset

$$M_0 = \{p \in M| \text{ Stab}(p) \text{ is non-trivial }\}.$$

Since the stabiliser of a point is a Lie Subgroup of G, define for each Lie subgroup $H \subset G$

$$M_H = \{p \in M| \text{ Stab}(p) = H\}.$$

We immediately see that

$$M_0 = \bigcup_{H \subset G} M_H.$$

We are very fortunate in this situation, as, following Berline, Getzler and Verne [8], we know that each M_H is a submanifold. The argument is illustrated as follows.

By averaging over G if necessary, we may assume that M possesses a G-invariant Riemannian metric.

Proposition 3.1.3 *A G-invariant Riemannian metric on M yields a G-invariant Levi-Civita connection.*

We can prove this last statement by showing that $\mathrm{Ad}(g)\nabla$ satisfies the definition of the Levi-Civita connection, and conclude that the two are equal, by the uniqueness of the Levi-Civita connection.

Proposition 3.1.4 *M_H is a submanifold of M.*

Proof
Let $p \in M_H$, then we have the isotropy representation $\rho : H \longrightarrow \mathrm{Aut}(\mathrm{T}_pM)$ given by the derivative of each $g \in H$ as a diffeomorphism of M, i.e.

$$\rho(g) = \mathrm{D}g_p.$$

Fix $\xi \in \mathfrak{h}$, $s \in \mathbb{R}$ and choose v in the fixed set of $\rho(\exp(s\xi))$. Consider the geodesic γ starting at p in the direction v. The geodesic equation is

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0.$$

Set $\tilde{\gamma} = \exp(s\xi)\gamma$, then

1. $\tilde{\gamma}(0) = \exp(s\xi)\gamma(0) = \exp(s\xi)p = p$;
2. $\dot{\tilde{\gamma}}(0) = \exp(s\xi)\gamma_*\dot{\gamma}(0) = \exp(s\xi)\gamma_* v = v$;
3. we have

$$\begin{aligned}\nabla_{\dot{\tilde{\gamma}}(t)}\dot{\tilde{\gamma}}(t) &= \exp(s\xi)^*\nabla_{\dot{\gamma}(t)} \exp(s\xi)_*\dot{\gamma}(t) \\ &= \exp(s\xi)_*\left(\exp(s\xi)^*\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\right) \\ &\quad \text{by the invariance of the Levi-Civita connection} \\ &= 0\end{aligned}$$

since γ satisfies the geodesic equation.

Hence M_H contains the geodesic γ. This is true for all $\xi \in \mathfrak{h}$,and for any geodesic γ with $\gamma(0) \in M_H$ and $\dot{\gamma}(0)$ tangent to M_H so M_H is a submanifold. ■

3.1.4 The Principle of Localisation

Here we follow the topological approach of Atiyah and Bott [5] as detailed by Michael Selby [28]. First, consider the inclusion of a manifold Y in the compact manifold X. Let $\pi : \nu Y \longrightarrow Y$ be the normal bundle of this inclusion and k the codimension of Y in X.

The equivariant Thom isomorphism says

$$\begin{aligned} H^q_T(Y) \longrightarrow H^{q+k}_{T,cpt}(\nu Y) &\cong H^{q+k}_T(X, X\backslash Y) \\ c \mapsto \pi^* c \wedge u & \end{aligned}$$

where u is the equivariant Thom class.

Now, we know that

$$H^\bullet_T(\mathrm{pt}) = H^\bullet_T = \mathbb{C}[\xi_1, \dots, \xi_l]$$

where l is the dimension of T. Define

$$H'^\bullet_T = \mathbb{C}(\xi_1, \dots, \xi_l)$$

and

$$H'^\bullet_T(X) = H^\bullet_T(X) \otimes_{H^\bullet_T} H'^\bullet_T.$$

Now suppose that Y is the set of T-fixed points of X. We have the following Lemma.

Lemma 3.1.5 (Selby [28]) *If T acts freely on X then $H'^{\bullet}_T(X) = 0$.*

Proof

Since T acts freely, we must have $H^{\bullet}_T(X) \cong H^{\bullet}(X/G)$. Hence $H^q_T(X) \cong 0$ for all sufficiently large q. Choose a non zero $\xi \in H^{\bullet}_T$. Then for each $\alpha \in H^{\bullet}_T(X)$, $\beta \in H'^{\bullet}_T$, we have

$$\alpha \otimes \beta = \xi^N \alpha \otimes \xi^{-N} \beta = 0$$

for sufficiently large N (the tensor product being over $H^{\bullet}_T$). Hence

$$\alpha \otimes \beta = 0$$

and since these elements generate $H'^{\bullet}_T(X)$, the result follows. ∎

Corollary 3.1.6 (Selby [28]) *In the long exact sequence of the pair $(X, X\backslash Y)$, the map*

$$H'_T(X, X\backslash Y) \longrightarrow H'_T(X)$$

is an isomorphism.

This follows immediately since T acts freely on $X\backslash Y$. Given a map $f : M \longrightarrow N$ between compact manifolds, we have the "umkehrungs" or pushforward homomorphism

$$f_* : H^q_T(M) \longrightarrow H^{q+n-m}_T(N)$$

(where m is the dimension of M and n of N), that satisfies

$$(f_*\alpha) \wedge \beta = f_*(\alpha \wedge f^*\beta), \tag{3.6}$$

for $\alpha \in H^{\bullet}(M)$, $\beta \in H^{\bullet}(N)$. In particular, if f is a fibration, then f_* is merely integration along the fibre.

Now, consider the sequence

$$Y \xhookrightarrow{\iota} X \xrightarrow{q} \{\mathrm{pt}\}.$$

Let $\alpha \in H^{\bullet}_T(X)$, then by (3.6)

$$\iota_*(\iota^*\alpha) = \alpha \wedge \iota_*(1)$$

where $1 \in H'_T(Y)$ is the formal unit. Hence $u = \iota_*(1)$ is the equivariant Thom class and is invertible being the composition

$$H'^{\bullet}_T(Y) \xrightarrow{\text{Thom}} H'^{\bullet+k}_T(X, X\backslash Y) \xrightarrow{\cong} H'^{\bullet+k}_T(X).$$

Recall that the Euler class is defined to be $e = \iota^* u = \iota^*\iota_*(1)$. In his thesis [28], Selby shows that e and u are both invertible in $H'^{\bullet}(X)$. This allows us to prove the following theorem.

Theorem 3.1.7 *Let the torus T act on the compact manifold M with fixed set M_0, then for any $\alpha \in H_T^\bullet(M)$*

$$\int_M \alpha = \int_{M_0} \frac{\iota^*\alpha}{e(\nu(M_0))}$$

where $\nu(M_0)$ is the normal bundle of M_0 in M

For the proof, again see [28] pp24-25.

3.1.5 Localisation for Manifolds With Boundary

Let M be a compact Riemannian manifold with boundary and a unitary action of S^1, and again let M_0 be the set of points fixed by every element of S^1

Lemma 3.1.8 *Let $\iota_\partial : \partial M_0 \hookrightarrow M_0$ be inclusion. If $\mathcal{V}(A;B)$ denotes the normal bundle of A in B and E the bundle of vectors normal to both M_0 and ∂M then*

$$\iota_\partial^* \mathcal{V}(M_0; M) = \mathcal{V}(\partial M_0, \partial M) \oplus E$$

Proof
We have for each $p \in \partial M_0$

$$(\mathrm{T}_p M_0 + \mathrm{T}_p \partial M) \oplus E_p = \mathrm{T}_p M$$

Hence

$$\begin{aligned} \mathcal{V}_p(M_0; M) &= \frac{\mathrm{T}_p M}{\mathrm{T}_p M_0} \\ &= \frac{(\mathrm{T}_p M_0 + \mathrm{T}_p \partial M) \oplus E_p}{\mathrm{T}_p M_0} \\ &= \frac{\mathrm{T}_p \partial M}{\mathrm{T}_p M_0 \cap \mathrm{T}_p \partial M} \oplus E_p \\ &= \frac{\mathrm{T}_p \partial M}{\mathrm{T}_p (M_0 \cap \partial M)} \oplus E_p \\ &= \frac{\mathrm{T}_p \partial M}{\mathrm{T}_p \partial M_0} \oplus E_p \\ &= \mathcal{V}_p(\partial M_0; \partial M) \oplus E_p. \end{aligned}$$

The result follows from this. ∎

Corollary 3.1.9 *If $M_0 \pitchfork \partial M$ then*

$$\mathcal{V}_p(M_0; M) = \mathcal{V}_p(\partial M_0; \partial M).$$

Proof
If $M_0 \pitchfork \partial M$ then $E = 0$. ■

Definition 3.1.10 *We shall say that a group G acts transversely on a manifold M with boundary if $M_0 \pitchfork \partial M$.*

Theorem 3.1.11 *Let M be a compact Riemannian manifold with boundary and let the circle S^1 act orthogonally on M with M_0 the set of points in M fixed by every element of S^1. Let α be an equivariantly closed form on M and E be the bundle of vectors normal to M_0 and ∂M, then*

$$\int_M \alpha(\xi) = \int_{M_0} \frac{\iota^*\alpha}{e(\xi)} - \int_{\partial M_0} \iota_\partial^* \left(\frac{\iota^*\alpha}{e(\xi)^2}\right) e(E)(\xi)$$

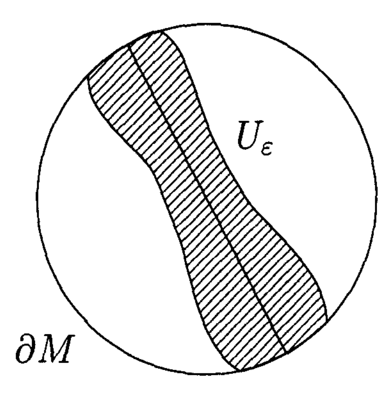
where $\iota : M_0 \hookrightarrow M$ and $e(\xi)$ is the equivariant Euler class of the normal bundle of M_0 within M.

Proof
Choose θ so that

$$\mathcal{L}_\xi \theta = 0$$

for each $\xi \in \mathrm{LieS}^1$ and that $\theta(X_\xi) = 1$.

$$\begin{aligned}
\int_M \alpha(\xi) &= \lim_{\varepsilon \longrightarrow 0} \int_{\partial(M \setminus U_\varepsilon)} \frac{\alpha(\xi)\theta}{\mathrm{d}_\mathfrak{g}\theta(\xi)} \\
&= \lim_{\varepsilon \longrightarrow 0} \int_{\partial M \setminus \partial U_\varepsilon} \frac{\alpha(\xi)\theta}{\mathrm{d}_\mathfrak{g}\theta(\xi)} - \lim_{\varepsilon \longrightarrow 0} \int_{\partial U_\varepsilon \setminus \partial M} \frac{\alpha(\xi)\theta}{\mathrm{d}_\mathfrak{g}\theta(\xi)}
\end{aligned}$$



Now,

$$-\lim_{\varepsilon \longrightarrow 0} \int_{\partial U_\varepsilon \setminus \partial M} \frac{\alpha(\xi)\theta}{\mathrm{d}_\mathfrak{g}\theta(\xi)} = \int_{M_0} \frac{\iota_0^*\alpha(\xi)}{e(\xi)}.$$

Also

$$\begin{aligned}
\lim_{\varepsilon\longrightarrow 0}\int_{\partial M\setminus\partial U_\varepsilon}\frac{\alpha(\xi)\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)} &= \lim_{\varepsilon\longrightarrow 0}\int_{\partial M\setminus\partial U_\varepsilon}\mathrm{d}_{\mathfrak{g}}\left(\frac{\alpha(\xi)\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)}\frac{\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)}\right)\\
&= \lim_{\varepsilon\longrightarrow 0}\int_{\partial(\partial M\setminus\partial U_\varepsilon)}\left(\frac{\alpha(\xi)\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)}\frac{\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)}\right)\\
&= -\lim_{\varepsilon\longrightarrow 0}\int_{\partial U_\varepsilon\cap\partial M}\left(\frac{\alpha(\xi)\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)}\frac{\theta}{\mathrm{d}_{\mathfrak{g}}\theta(\xi)}\right)\\
&= -\int_{\partial M_0}\iota_\partial^*\left(\frac{\iota^*\alpha}{e(\xi)}\right)\frac{1}{e(\mathcal{V}(\partial M_0;\partial M))(\xi)}\\
&= -\int_{\partial M_0}\iota_\partial^*\left(\frac{\iota^*\alpha}{e(\xi)}\right)\frac{e(E)(\xi)}{e(\mathcal{V}(\partial M_0;\partial M)\oplus E)(\xi)}\\
&= -\int_{\partial M_0}\iota_\partial^*\left(\frac{\iota^*\alpha}{e(\xi)^2}\right)e(E)(\xi)
\end{aligned}$$

using Lemma 3.1.8 and functoriality of the Euler class.

Hence we have,

$$\int_M\alpha(\xi)=\int_{M_0}\frac{\iota_0\alpha(\xi)}{e(\xi)}-\int_{\partial M_0}\iota_\partial^*\left(\frac{\iota^*\alpha}{e(\xi)^2}\right)e(E)(\xi)$$

as desired. ∎

Corollary 3.1.12 *If* S^1 *acts transversely we have,*

$$\int_M\alpha(\xi)=\int_{M_0}\frac{\iota_0\alpha(\xi)}{e(\xi)}-\int_{\partial M_0}\iota_\partial^*\left(\frac{\iota^*\alpha}{e(\xi)^2}\right).$$

Corollary 3.1.13 *Let* M *be a compact manifold with boundary with smooth transverse action of* S^1 *and* $\alpha\in\Omega^\bullet_{S^1}(M)$. *Then*

$$\int_M\mathrm{d}_{\mathfrak{g}}\alpha=\int_{M_0}\frac{\iota^*\mathrm{d}_{\mathfrak{g}}\alpha}{e}.$$

Proof
Follows from Theorem 3.1.11, the boundary term vanishing due to exactness of the integrand. ∎

Corollary 3.1.14 *Let* M *be a non-compact manifold with smooth action of* S^1 *and* $\alpha\in\Omega^\bullet_{S^1}(M)$ *be compactly supported. Then*

$$\int_M\alpha=\int_{M_0}\frac{\iota^*\alpha}{e}.$$

Remark
Notice now that we can remove the issue of compactness in Corollary 3.1.13. We can take a sequence of compact submanifolds with boundary which exhausts the manifold and prove the identity on all of these. Thus the question of existence of the integral is reduced merely to the question as to whether the form $\mathrm{d}_{\mathfrak{g}}\alpha$ is integrable.

3.2 Equivariant Cohomology of Symplectic Manifolds

3.2.1 Preliminaries

Let (M, ω) be a compact symplectic manifold with a Hamiltonian action of the compact Lie group G. Let $\mu : M \longrightarrow \mathfrak{g}^*$ be the moment map of this action, and assume that 0 is a regular value of μ so that $\mu^{-1}(0)$ is a manifold and hence so is the Marsden-Weinstein quotient $M/\!\!/G = \mu^{-1}(0)/G$. Our goal is to relate integrals (i.e cohomology) over $M/\!\!/G$ to integrals over M.

Let $N = \mu^{-1}(0)$, then we have the principal bundle

$$\pi : N \longrightarrow \mathcal{M} = N/G.$$

Let $\xi_1 \dots \xi_s$ be an orthonormal basis of G and $\theta = \theta_i \xi_i$ be a connection on $N \longrightarrow \mathcal{M}$, then

$$\Omega = \bigwedge_{i=1}^{s} \theta_i$$

is a volume form on the fibre of N, hence

$$\pi_* \Omega = \mathrm{vol} G.$$

The main result is

Theorem 3.2.1

$$\int_{\mathcal{M}} e^{i\omega_0} \quad = \quad \lim_{t \longrightarrow \infty} \frac{(i)^{s^2}}{(2\pi)^s \, \mathrm{vol} G} \int_{\mathfrak{g}} [dy] e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega + i\{\mu, y\}}$$

where at all times we are using a version of the Berezin integral formalism, that the integral of a k-form over a l-dimensional sub-manifold is zero if $k \neq l$.

3.2.2 The Kirwan Map

Let M be a compact symplectic manifold and G a compact connected Lie Group which acts in a Hamiltonian fashion on M, and let μ be a moment map. Suppose that 0 is a regular value of μ In [19], Frances Kirwan details a method by which we can show that there is a surjective map

$$\kappa : \mathrm{H}^\bullet_G(M; \mathbb{C}) \longrightarrow \mathrm{H}^\bullet(M/\!\!/G; \mathbb{C})$$

which is given as $\kappa = \iota^* (p^*)^{-1}$ where $p : \mu^{-1}(0) \longrightarrow M/\!\!/G$ is the quotient map with

$$p^* : \mathrm{H}^\bullet(M/\!\!/G) \longrightarrow \mathrm{H}^\bullet_G(\mu^{-1}(0))$$

being the isomorphism, since G acts locally freely on $\mu^{-1}(0)$, and $\iota : \mu^{-1}(0) \hookrightarrow M$ inclusion. This is proved using the Morse theory of the function $f = |\mu|^2$ using the Killing norm on $\mathfrak{g}^*$.

We should like to obtain a similar result for the boundaries of symplectic manifolds, and later consider hyperKähler manifolds and boundaries of hyperKähler manifolds.

The function f gives us a stratification of M, that is a finite collection of subspaces S_α, such that

1. $$M = \bigcup_{\alpha \in A} S_\alpha,$$

2. there is a partial ordering $>$ on A such that
$$\overline{S_\alpha} \subseteq \bigcup_{\gamma \geq \alpha} S_\gamma.$$

Let $M_\alpha = \bigcup_{\alpha \geq \gamma} S_\gamma$. We have the result

Lemma 3.2.2 ([19], Lemma 2.18 pp33-34) *Suppose $\{S_\gamma | \gamma \in A\}$ is a smooth G-invariant stratification of M such that for each $\alpha \in A$, the equivariant Euler class of the normal bundle to S_α in M, is not a zero divisor in*
$$\mathrm{H}^\bullet_G(S_\alpha; \mathbb{C}).$$
Then the inclusion
$$\iota : M_\alpha \backslash S_\alpha \hookrightarrow M_\alpha$$
induces a surjection
$$\iota^* : \mathrm{H}^\bullet_G(M_\alpha) \longrightarrow \mathrm{H}^\bullet_G(M_\alpha \backslash S_\alpha).$$

For the proof see [19]. For symplectic manifolds with a Hamiltonian action of S^1, we know by Proposition 3.1.4, that not only is the fixed set a submanifold, but its Euler class is not a zero divisor! Kirwan also proves that the stratifications obtained by minimally degenerate functions are just as good, and that moment maps are minimally degenerate and the stratifications satisfy the hypothesis in Theorem 3.2.2.

3.2.3 The Symplectic Structure near $\mu^{-1}(0)$

Guillemin and Sternberg in [16] give a proof of the coisotropic embedding theorem which allows us to describe the structure of M in a neighbourhood of N. We state it in the following form

Theorem 3.2.3 (The Coisotropic Embedding Theorem) *(Guillemin and Sternberg [16] p315) Given a symplectic manifold (B, ω_0) and a principal G-bundle $\pi : P \longrightarrow B$ then there is a unique [1] symplectic manifold (M, ω) into which P embeds as a coisotropic submanifold and the restriction of ω to P is precisely $\pi^* \omega_0$.*

[1] upto symplectomorphism

The main corollary to this is the following

Corollary 3.2.4 ([18]) *Given a compact symplectic manifold (M, ω) with a Hamiltonian G-action and moment map $\mu : M \longrightarrow \mathfrak{g}^*$ with 0 a regular value and $N = \mu^{-1}(0)$, there is a neighbourhood U of N and a diffeomorphism $\Phi : N \times \overline{B_h(0; \mathfrak{g}^*)} \longrightarrow U$ such that*

$$\Phi^*\omega = \pi^*\omega_0 + \mathrm{d}\tau$$

where $\tau_{(p,\phi)} = \{\phi, \theta_p\}$, where θ is a connection on $\pi : N \longrightarrow M/\!/G$.

Corollary 3.2.4 tells us that functions and forms supported on a sufficiently small neighbourhood of N can be replaced by functions and forms supported on a sufficiently small neighbourhood of $N \times \{0\}$ in $N \times \mathfrak{g}^*$. This will prove very useful when we come to consider equivariant integration.

3.2.4 Equivariant Integration

Theorem 3.2.1 was really introduced by Witten in [29] and the technique that we use to prove this theorem is by Jeffries and Kirwan in [18] although they go further and produce some results proving a localisation formula.

We need to set up some theory of Gaussian integrals and Fourier analysis on Lie algebras. First we have a technical result which we will need.

Proposition 3.2.5 *Let U and V be s-dimensional manifolds.If $\{a_j\}_{j=1}^s \subset \Omega^1(U)$ and $\{b_j\}_{j=1}^s \subset \Omega^1(V)$ then*

$$e_{[2s]}^{\left(i\sum_{j=0}^s a_j \wedge b_j\right)} = i^{s^2}[a] \wedge [b]$$

where $[a] = \bigwedge_{i=1}^s a_i$.

Proof
This is a simple calculation:

$$\begin{aligned}
e_{[2s]}^{\left(i\sum_{j=0}^s a_j \wedge b_j\right)} &= \frac{1}{s!}\left(i\sum_{j=0}^s a_j \wedge b_j\right)^s \\
&= \frac{i^s}{s!}\left[s! \bigwedge_{j=1}^s a_j \wedge b_j\right] \\
&= i^s(-1)^{\frac{s}{2}(s-1)}[a] \wedge [b] \\
&= i^{s^2}[a] \wedge [b].
\end{aligned}$$

■

We have a general Gaussian integral formula, the proof of which is a standard argument using the method of "Completing the square":

$$\int_{\mathfrak{g}}[\mathrm{d}x]e^{\left(-a|x|^2+b\langle x,y\rangle\right)} = \left(\sqrt{\frac{\pi}{a}}\right)^{s} e^{\left(\frac{b^2}{4a}|y|^2\right)}. \tag{3.7}$$

Putting $b=i$ and $a=1/4t$ into this, we have

$$\delta(x) = \lim_{t\longrightarrow\infty}\left(\sqrt{\frac{t}{\pi}}\right)^{s} e^{\left(-t|x|^2\right)} = \lim_{t\longrightarrow\infty}\left(\frac{1}{2\pi}\right)^{s}\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}+i\langle x,y\rangle\right)}. \tag{3.8}$$

Proof of 3.2.1

(Following [18]) We put a connection on N (or a metric on M) with connection form θ. The generalised form

$$\lim_{t\longrightarrow\infty}\int_{\mathfrak{g}} e^{\left(\frac{-|y|^2}{4t}\right)} e^{i\omega+i\{\mu,y\}}$$

on M is supported on $N=\mu^{-1}(0)$ by (3.8), hence by Corollary 3.2.4

$$\begin{aligned}
&\lim_{t\longrightarrow\infty}\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}\right)}\int_M e^{i\omega+i\{\mu,y\}}\\
=\ &\lim_{t\longrightarrow\infty}\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}\right)}\int_{N\times\mathfrak{g}^*} e^{i\pi^*\omega_0+\{\mathrm{d}z,\theta\}+i\{z,\mathrm{d}\theta\}+i\{z,y\}}\\
=\ &\lim_{t\longrightarrow\infty} i^{s^2}\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}\right)}\int_{N\times\mathfrak{g}^*} e^{i\pi^*\omega_0+i\{z,\mathrm{d}\theta\}+i\{z,y\}}[\mathrm{d}z]\Omega\\
=\ &\lim_{t\longrightarrow\infty} i^{s^2}\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}\right)}\int_{\mathfrak{g}^*} e^{i\{z,y\}}[\mathrm{d}z]\int_N e^{i\pi^*\omega_0+i\{z,\mathrm{d}\theta\}}\Omega\\
=\ &\lim_{t\longrightarrow\infty} i^{s^2}(-1)^{s^2}\int_{\mathfrak{g}^*} e^{i\{z,y\}}[\mathrm{d}z]\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}\right)}\int_N e^{i\pi^*\omega_0+i\{z,\mathrm{d}\theta\}}\Omega\\
=\ &i^{s^2}(-1)^{s^2}(2\pi)^s\int_{\mathfrak{g}^*}\delta(z)[\mathrm{d}z]\int_N e^{i\pi^*\omega_0+i\{z,\mathrm{d}\theta\}}\Omega\\
=\ &(-i)^{s^2}(2\pi)^s\int_N e^{i\pi^*\omega_0}\Omega\\
=\ &(-i)^{s^2}(2\pi)^s\mathrm{vol}G\int_{\mathcal{M}} e^{i\omega_0}.
\end{aligned}$$

■

We now obtain some useful formula for the integration of general forms on $\mathcal{M}$.

Theorem 3.2.6 *Let $\eta\in\Omega^\bullet_G(M)$ be equivariantly closed and have representative $\eta_0\in\Omega^\bullet(\mathcal{M})$. If M has no boundary then*

$$\int_{\mathcal{M}} e^{i\omega_0}\eta_0 = \lim_{t\longrightarrow\infty}\frac{(i)^{s^2}}{(2\pi)^s\,\mathrm{vol}G}\int_{\mathfrak{g}}[\mathrm{d}y]e^{\left(\frac{-|y|^2}{4t}\right)}\int_M e^{i\omega+i\{\mu,y\}}\eta(y).$$

Proof

The proof of this fact is very similar to the proof of Theorem 3.2.1, but there are a few technicalities to overcome first.

Suppose that $\eta = \sum_I \eta_I \phi_I$ where I is a multi-index, $\{\phi_i\}_{i=1}^s$ is a basis for $\mathfrak{g}^*$ and $\eta_I \in \Omega^\bullet(M)^G$. We have

$$
\begin{aligned}
&\int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega + i\{\mu, y\}} \eta(y) \\
=\ & \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega} \int_{\mathfrak{g}^*} [\mathrm{d}\phi] e^{i\{\phi, y\}} \delta(\phi - \mu) \eta(y) \\
=\ & \sum_I \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega} \int_{\mathfrak{g}^*} [\mathrm{d}\phi] e^{i\{\phi, y\}} \delta(\phi - \mu) \eta_I \phi^I(y) \\
=\ & (-1)^{s^2} \sum_I (-i)^{|I|} \int_M e^{i\omega} \eta_I \int_{\mathfrak{g}^*} [\mathrm{d}\phi] \delta(\phi - \mu) \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} e^{i\{\phi, y\}} i^{|I|} y_I.
\end{aligned}
$$

Now, notice that the last integral is a derivative of the Fourier transform of

$$y \mapsto e^{\left(\frac{-|y|^2}{4t}\right)}$$

which in the limit as $t \longrightarrow \infty$ is the delta-function $\phi \mapsto \delta(\phi)$. Hence the generalised function

$$\delta(\phi - \mu) \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} e^{i\{\phi, y\}} i^{|I|} y_I$$

is supported only on N, which means that by Corollary 3.2.4, we can replace M with $N \times \mathfrak{g}^*$ in the original integral.

Now by Theorem 3.1.2 we have

$$\iota^* \eta = \mathrm{d}_{\mathfrak{g}} \beta + \pi^* \eta_0$$

for some $\beta \in \Omega_G^\bullet(M)$, where $\iota : \mu^{-1}(0) \hookrightarrow M$ is the inclusion. Hence

$$
\begin{aligned}
\int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega + i\{\mu, y\}} \eta \ &= \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_{N \times \mathfrak{g}^*} e^{i\omega + i\{\mu, y\}} \mathrm{d}_{\mathfrak{g}} \beta(y) \\
&+ \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_{N \times \mathfrak{g}^*} e^{i\omega + i\{\mu, y\}} \pi^* \eta_0.
\end{aligned}
$$

Now let us examine the first term here. From the above argument

$$\begin{aligned}
& \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_{N\times\mathfrak{g}^*} e^{i\omega + i\{\mu,y\}} \mathrm{d}_{\mathfrak{g}}\beta(y) \\
= & \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_{N\times\mathfrak{g}^*} \mathrm{d}_{\mathfrak{g}} \left(e^{i\omega + i\mu(y)}\beta\right)(y) \\
& \text{since } \omega + \mu(y) \text{ is an equivariantly closed form,} \\
= & \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_{N\times\mathfrak{g}^*} \mathrm{d} \left(e^{i\omega + i\mu(y)}\beta\right)(y) \\
- & \int_{\mathfrak{g}} [\mathrm{d}y] e^{\left(\frac{-|y|^2}{4t}\right)} \int_{N\times\mathfrak{g}^*} X_y \lrcorner \left(e^{i\omega + i\mu(y)}\beta\right)(y) \\
= & \; 0.
\end{aligned}$$

The last term here vanishes since the integrand does not contain differential forms of top degree, the first by Stokes' theorem.

The rest of the proof now follows that of Theorem 3.2.1. ■

Remark
Using Theorem 3.2.2, we find that for every $\eta_0 \in \Omega^\bullet(\mathcal{M})$, there is a $\eta \in \Omega^\bullet_G(M)$ such that

$$\iota^*\eta = \pi^*\eta_0 + \mathrm{d}_{\mathfrak{g}}\beta,$$

where $\iota : \mu^{-1}(0) \hookrightarrow M$ is the inclusion and $\pi : \mu^{-1}(0) \longrightarrow \mathcal{M}$ is the quotient map.

We will simplify matters by following Witten's notation, and write

$$\oint_M \alpha(\xi) = \frac{1}{\operatorname{vol}(G)} \lim_{t\longrightarrow\infty} \int_{\mathfrak{g}} \mathrm{d}\xi e^{-\frac{|\xi|^2}{4t}} \int_M \alpha(\xi).$$

In this notation we may rewrite the result of Theorem 3.2.6 as

$$\int_{\mathcal{M}} e^{i\omega_0}\eta_0 = \frac{(i)^{s^2}}{(2\pi)^s} \oint_M e^{i\omega + i\{\mu,y\}}\eta(y).$$

3.2.5 Localisation and Residues

First we observe that in the case of a circle action, by Theorem 3.2.6 we have a Fourier transform

$$\begin{aligned}
\int_{\mathcal{M}} e^{i\omega_0}\eta_0 &= \frac{i}{(2\pi)} \oint_M e^{i\omega + i\{\mu,y\}}\eta(y). \\
&= \frac{i}{\sqrt{2\pi}\operatorname{vol} S^1} \mathbf{F}\left(z \mapsto \int_M e^{i\omega + i\{\mu,z\}}\eta(z)\right)(0),
\end{aligned}$$

where $\mathbf{F}$ denotes Fourier transform.

We also have a rather interesting fact. Let M be a Riemannian manifold, V a vector space and $f : M \longrightarrow V$ a smooth map. Suppose that 0 and v are regular values of f such that that the line segment $[0, v]$ is a set of regular values of f. For any $t \in [0, v]$ set

$$M_t = f^{-1}(tv).$$

Now we have a way of relating M_s and M_t. Let X be the vector field on M normal to $\mu^{-1}(tv)$ for all t defined by

$$\mathrm{D}f_p X(p) = \frac{\partial}{\partial t}(f(p))$$

where $\frac{\partial}{\partial t}$ is the unit tangent vector on $[0, v]$ pointing to v. This is well defined since $\mathrm{D}f_p X(p)$ fails to be an isomorphism of the normal space to M_t at p with V only on critical points which do not lie in $f^{-1}([0, v])$. The flow of this vector field gives us a diffeomorphism $\sigma_t : M \longrightarrow M$ with the property that

$$\sigma_t : M_s \longrightarrow M_{tv+s}$$

whenever $s + tv \in [0, v]$. So now we know that the diagram

$$\begin{array}{ccccc} & \iota_s^* & & & \\ \mathrm{H}^\bullet(M) & \longrightarrow & \mathrm{H}^\bullet(M_s) & & \\ & \searrow & \downarrow & & \sigma_s^* \\ & \iota_0^* & \mathrm{H}^\bullet(M_a) & & \end{array}$$

(where $\iota_t : M_t \longrightarrow M$ is inclusion) commutes since we have an obvious homotopy between $\iota_s \circ \sigma_s$ and ι_0. If $c \in \mathrm{H}^\bullet(M)$, then for all $t \in [0, 1]$

Theorem 3.2.7

$$\int_{M_0} \iota_0^* c = \int_{M_t} \iota_t^* c.$$

We now apply the theory above to $f = \mu$ and $v = \zeta$ If $\eta \in \Omega^\bullet_{\mathrm{S}^1}(M)$ and $\iota_t : \mu^{-1}(t\zeta) \hookrightarrow M$ then we have $\eta_0 \in H^\bullet(\mu^{-1}(0)/\mathrm{S}^1)$ such that

$$\iota_0^* \eta = \pi^* \eta_0 + \mathrm{d}_{\mathfrak{g}} \beta.$$

So we have

Theorem 3.2.8 *For sufficiently small $\zeta \in \mathbb{R}$*

$$\begin{aligned} \int_{\mathcal{M}} e^{i\omega_0} \eta_0 &= \lim_{t \longrightarrow \infty} \frac{(i)}{(2\pi)\,\mathrm{vol}\mathrm{S}^1} \int_{\mathbb{R}} \mathrm{d}y e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega + i\{\mu, y\}} \eta(y) \\ &= \lim_{t \longrightarrow \infty} \frac{(i)}{(2\pi)\,\mathrm{vol}\mathrm{S}^1} \int_{\mathbb{R}} \mathrm{d}y e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega + i\{(\mu - \zeta), y\}} \eta(y) \\ &= \lim_{t \longrightarrow \infty} \frac{(i)}{(\sqrt{2\pi})\,\mathrm{vol}\mathrm{S}^1} \mathbf{F}\left(y \mapsto e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega + i\{\mu, y\}} \eta(y)\right)(\zeta). \end{aligned}$$

Proof

$$\begin{aligned}
\int_{\mu^{-1}(0)/S^1} \eta_0 &= \int_{\mu^{-1}(0)} \pi^*\eta_0 \wedge \iota_0^*\Theta \\
&\quad \Theta \text{ being the fibre volume on the principal bundle} \\
&\quad \mu^{-1}([0,\zeta]) \longrightarrow \mu^{-1}([0,\zeta])/S^1 \\
&= \int_{\mu^{-1}(0)} \iota_0^*\eta \wedge \iota_0^*\Theta \\
&= \int_{\mu^{-1}(\zeta)} \iota_1^*\eta \wedge \iota_1^*\Theta \\
&\quad \text{by Theorem 3.2.7} \\
&= \lim_{t\longrightarrow\infty} \frac{(i)}{(2\pi)\,\mathrm{vol}S^1} \int_{\mathbb{R}} \mathrm{d}y e^{\left(\frac{-|y|^2}{4t}\right)} \int_M e^{i\omega+i\{(\mu-\zeta),y\}}\eta(y)
\end{aligned}$$

where the last equality follows by the proof of Theorem 3.2.6. ∎

So this last result means that

$$\frac{i}{\sqrt{2\pi}}\mathbf{F}\left(z \mapsto \int_M e^{i\omega+i\{\mu,z\}}\eta(z)\right)$$

is smooth on a neighbourhood of 0. But by the localisation theorem (Theorem 3.1.7), we know that

$$\int_M e^{i\omega+i\{\mu,z\}}\eta(z) = \int_{M_0} \frac{\iota_{M_0}^*\left(e^{i\omega+i\{\mu,z\}}\eta(z)\right)}{e(z)}$$

where $e(z)$ is the Euler class of the normal bundle of M_0 in M. Thus, by Proposition 8.7 of [18]

$$\begin{aligned}
&\mathbf{F}\left(z \mapsto \int_M e^{i\omega+i\{\mu,z\}}\eta(z)\right)(0) \\
&= \lim_{\varepsilon\longrightarrow 0+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}-i\xi} \mathbf{F}(\chi)(\varepsilon\psi)\mathbf{F}^2\left(z \mapsto \int_{M_0} \frac{\iota_{M_0}^* e^{i\omega+i\{\mu,z\}}\eta(z)}{e(z)}\right)(\psi)\mathrm{d}\psi \\
&= \lim_{\varepsilon\longrightarrow 0+} \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}-i\xi} \mathbf{F}(\chi)(\varepsilon\psi) \int_{M_0} \frac{\iota_{M_0}^*\left(e^{i\omega+i\{\mu,-\psi\}}\eta(-\psi)\right)}{e(-\psi)}\mathrm{d}\psi
\end{aligned}$$

where χ is a smooth positive function on $\mathbb{R}$ with compact support and $\xi \in \mathbb{R}$. Professors Kirwan and Jeffries show that because

$$\mathbf{F}\left(z \mapsto \int_M e^{i\omega+i\{\mu,z\}}\eta(z)\right)$$

is smooth at 0, the latter integral is independent of χ and ξ. Now we may bring to bear the standard theory of complex analysis. We know that $\mathbf{F}(\chi)$ is entire when extended to $\mathbb{C}$ because χ is smooth and compactly supported on $\mathbb{R}$ (Proposition 8.4 of [18]), we also know that μ is constant and non-zero on M_0 since 0 is a regular value of μ. Hence using the Residue theorem from complex analysis

$$\begin{aligned}
&\lim_{\varepsilon \longrightarrow 0_+} \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}-i\xi} \mathbf{F}(\chi)(\varepsilon\psi) \int_{M_0} \frac{\iota^*_{M_0}\left(e^{i\omega+i\{\mu,-\psi\}}\eta(-\psi)\right)}{e(-\psi)} \mathrm{d}\psi \\
= &-\frac{i}{\sqrt{2\pi}} \mathrm{Coeff}_{y^{-1}} \int_{M_0} \frac{\iota^*_{M_0} e^{i\omega+i\{\mu,y\}}\eta(y)}{e(y)}
\end{aligned}$$

where we always take the Laurent expansion of the integrand at $y = 0$. Thus we end up with the following theorem.

Theorem 3.2.9 *Let M be compact symplectic manifold with Hamiltonian action of the circle such that 0 is a regular value of the moment map μ .Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be inclusion. If $\eta \in \Omega_{\mathrm{S}^1}(M)$ is equivariantly closed and satisfies*

$$\iota^*\eta(y) = \pi^*\eta_0 + \mathrm{d}_{\mathfrak{g}}\beta(y)$$

for some closed $\eta_0 \in \Omega(M/\!\!/\mathrm{S}^1)$ and $\beta \in \Omega_{\mathrm{S}^1}(M)$, then

$$\begin{aligned}
\int_{\mathcal{M}} e^{i\omega_0}\eta_0 &= \frac{i}{(2\pi)} \oint_M e^{i\omega+i\{\mu,y\}}\eta(y) \\
&= \frac{1}{2\pi \mathrm{vol}\,\mathrm{S}^1} \mathrm{Coeff}_{y^{-1}} \int_{M_0} \frac{\iota^*_{M_0}\left(e^{i\omega+i\{\mu,y\}}\eta(y)\right)}{e(y)}
\end{aligned}$$

where we always take the Laurent expansion of the integrand at $y = 0$.

Corollary 3.2.10 *Let M be compact symplectic manifold with Hamiltonian action of the torus $\mathbb{T}^k$ such that $0 \in \mathrm{Lie}(\mathbb{T}^k)$ is a regular value of the moment map μ. Choose a splitting of $\mathbb{T}^k$ into the product of circles Let $\iota : \mu^{-1}(0) \hookrightarrow M$ be inclusion. If $\eta \in \Omega_{\mathbb{T}^k}(M)$ is equivariantly closed satisfies*

$$\iota^*\eta(y_1, \dots, y_k) = \pi^*\eta_0 + \mathrm{d}_{\mathfrak{g}}\beta(y_1, \dots, y_k)$$

for some closed $\eta_0 \in \Omega(M/\!\!/\mathrm{S}^1)$ and $\beta \in \Omega_{\mathrm{S}^1}(M)$, then

$$\begin{aligned}
\int_{\mathcal{M}} e^{i\omega_0}\eta_0 &= \left(\frac{i}{2\pi}\right)^k \oint_M e^{i\omega+i\{\mu,\underline{y}\}}\eta(\underline{y}) \\
&= \left(\frac{1}{2\pi}\right)^k \frac{1}{\mathrm{vol}\,\mathbb{T}^k} \mathrm{Coeff}_{y_1^{-1},\dots,y_k^{-1}} \int_{M_0} \frac{\iota^*_{M_0}\left(e^{i\omega+i\{\mu,\underline{y}\}}\eta(\underline{y})\right)}{e(\underline{y})}
\end{aligned}$$

where we always take the Laurent expansion of the integrand at $\underline{y} = 0$.

3.3 Equivariant Integrals in HyperKähler Reduction

3.3.1 Preliminaries

Recall that a hyperKähler manifold $(M, \vec{\omega})$ is a $4k$-dimensional Riemannian manifold with three symplectic forms which form the components of the $\mathfrak{sp}(1)$-valued 2-form

$$\vec{\omega} = \omega_{\mathbb{i}} \otimes \mathbb{i} + \omega_{\mathbb{j}} \otimes \mathbb{j} + \omega_{\mathbb{k}} \otimes \mathbb{k}$$

whose associated complex structures (resp $(\mathbb{i}, \mathbb{j}, \mathbb{k})$) obey

$$\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = \mathbb{i}\mathbb{j}\mathbb{k} = -\mathbb{1}.$$

Let M be compact and G act on M with a tri-Hamiltonian action, that is, the action is Hamiltonian with respect to each of the symplectic forms. Then we have a threefold moment map

$$\vec{\mu} = (\mu_{\mathbb{i}}, \mu_{\mathbb{j}}, \mu_{\mathbb{k}}) : M \longrightarrow \mathfrak{g} \otimes_{\mathbb{R}} \Im\mathbb{H}$$

such that

$$\{\mathrm{d}\vec{\mu}(X), \xi\} = \vec{\omega}(X_\xi, X)$$

for each $\xi \in \mathfrak{g}, X \in \Gamma(\mathrm{T}M)$. For an element $\vec{a} = a_0 + a_1\mathbb{i} + a_2\mathbb{j} + a_3\mathbb{k} \in \mathbb{H}$ and $X \in \mathrm{T}_pM$, we set

$$\vec{a}X = a_0X + a_1\mathbb{i}X + a_2\mathbb{j}X + a_3\mathbb{k}X.$$

If 0 is a regular value of $\vec{\mu}$, we can form the hyperKähler reduction

$$\mathcal{M} = \vec{\mu}^{-1}(0)/G = M/\!/\!/\!/G.$$

Now we can regard this reduction in several ways:

1. as the quotient of $N_0 = \vec{\mu}^{-1}(0)$ by G,
2. if we fix a complex structure $\mathbb{i}$ then set $\omega_{\mathbb{C}} = \omega_{\mathbb{j}} + i\omega_{\mathbb{k}}$ and $\mu_{\mathbb{C}} = \mu_{\mathbb{j}} + i\mu_{\mathbb{k}}$, then we can view $M/\!/\!/\!/G$ as $N_{\mathbb{C}}/\!/G$ with respect to the Kähler form $\omega_{\mathbb{i}}$ where $N_{\mathbb{C}} = {\mu_{\mathbb{C}}}^{-1}(0)$, a Kähler reduction,
3. fix a complex structure $\mathbb{i}$ and take the GIT symplectic reduction of M by the complex group $G_{\mathbb{C}}$.

Essentially we'd like to use the first point of view. It is independent of any choices and the various spaces and groups involved will be compact, a fact that the other choices do not share.

We do have *not* have a hyperKähler version of the Coisotropic embedding theorem due to lack of a hyper-Darboux theorem. However, we can make some attempt at looking at the hyperKähler structure near $\vec{\mu}^{-1}(0)$ in the simpler case of a torus action. 3.2.4

Lemma 3.3.1 *Let $(M, \langle\cdot,\cdot\rangle, \vec{\omega})$ be a hyperKähler manifold with a tri-Hamiltonian action of the k-dimensional torus G, and moment map $\vec{\mu} : M \longrightarrow \mathfrak{g}^* \otimes \mathfrak{sp}(1) \cong \mathfrak{g} \otimes \Im\mathbb{H}$. Suppose that 0 is a regular value of $\vec{\mu}$, so that $N = \mu^{-1}(0)$ is a submanifold. Then there are k differential 1-forms $\phi^\alpha \in \Omega^1(M; \mathbb{H})$ such that on a sufficiently small tubular neighbourhood of N, we have*

$$\vec{\omega} = \vec{\omega}' - \phi^\alpha \wedge \overline{\phi^\alpha}$$

where $\vec{\omega}'$ is closed and restricts on N to $\pi^*\vec{\omega}_0$, the pullback under the projection of the hyperKähler structure $\vec{\omega}_0$ of $M/\!/\!/\!/G$. As a result $\phi^\alpha \wedge \overline{\phi^\alpha}$ is also closed.

Proof
Note that in a small tubular neighbourhood U around N in M, $\mathrm{D}\vec{\mu}_p$ is surjective for all $p \in U$. So restricted to U

$$\mathrm{T}M \cong \ker \mathrm{d}\vec{\mu} \oplus (\ker \mathrm{d}\vec{\mu})^{\perp}$$

Now, examining $\ker \mathrm{d}\vec{\mu}$, we notice that, because the group acting is a torus, we have for all $p \in U$, $\xi, \eta \in \mathfrak{g}$

$$\begin{aligned} \vec{\omega}\left(X_\xi(p), X_\eta(p)\right) &= \{\vec{\mu}(p), [\xi, \eta]\} \\ &= 0 \end{aligned} \tag{3.9}$$

where $\{\cdot,\cdot\}$ denotes the invariant inner product on $\mathfrak{g}$. Denote the sub-bundle of $\ker \mathrm{d}\vec{\mu}$ by V which form the vertical vector fields over M/G, and set $H = \ker \mathrm{d}\vec{\mu} \cap (V)^{\perp}$. Now, let $\{X_\alpha\}_{\alpha=1}^{k}$ be an orthonormal frame of V over a (possibly smaller) U. Then

$$\mathrm{T}M = H \oplus V \oplus V_{\Im\mathbb{H}}$$

where $V_{\Im\mathbb{H}} = \mathrm{Span}\{qX_\alpha | q \in \Im\mathbb{H}\}_{\alpha=1}^{k}$. Now we can set

$$\begin{aligned} \phi_0^\alpha &= \frac{1}{\sqrt{2}}\langle X_\alpha, \cdot\rangle \\ \phi_1^\alpha &= \frac{1}{\sqrt{2}}\langle \mathbb{i} X_\alpha, \cdot\rangle \\ \phi_2^\alpha &= \frac{1}{\sqrt{2}}\langle \mathbb{j} X_\alpha, \cdot\rangle \\ \phi_3^\alpha &= \frac{1}{\sqrt{2}}\langle \mathbb{k} X_\alpha, \cdot\rangle \end{aligned}$$

and

$$\phi^\alpha = \phi_0^\alpha + \mathbb{i}\phi_1^\alpha + \mathbb{j}\phi_2^\alpha + \mathbb{k}\phi_3^\alpha.$$

Now setting $q_0 = 1, q_1 = \mathbb{i}, q_2 = \mathbb{j}, q_3 = \mathbb{k}$, we have for all α, β and $i, j \in \{0, 1, 2, 3\}$

$$\begin{aligned} \langle q_i X_\beta, q_j X_\gamma\rangle &= \langle q_j^* q_i X_\beta, X_\gamma\rangle \\ &= \begin{cases} \omega_{q_j^* q_i}(X_\beta, X_\gamma) & i \neq j \\ \langle X_\beta, X_\gamma\rangle & i = j \end{cases} \\ &= \delta_{ij}\delta_{\beta\gamma}, \end{aligned}$$

since $\vec{\omega}$ vanishes on V by (3.9). Hence the vector fields $\{q_i X_\beta\}_{\alpha=1}^{k}{}_{i=0}^{3}$ form an orthonormal frame for $H^{\perp}$ on U. Using this we can show that

$$\vec{\omega}(q_j X_\beta, q_k X_\gamma) = (\varepsilon_{ijk} + \delta_{0j}\delta_{ik} - \delta_{0k}\delta_{jk})\,\delta_{\beta\gamma} q_i$$

where

$$\varepsilon_{ijk} = \begin{cases} 1 & (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise.} \end{cases}$$

It can be readily checked that

$$\phi^\alpha \wedge \overline{\phi^\alpha}(q_j X_\beta, q_k X_\gamma) = -\left(\varepsilon_{ijk} + \delta_{0j}\delta_{ik} - \delta_{0k}\delta_{jk}\right)\delta_{\beta\gamma} q_i.$$

Now let $\vec{\omega}' = \vec{\omega} + \phi^\alpha \wedge \overline{\phi^\alpha}$. Now we know by construction that for $v \in H, X \in H^\perp$ we have $\vec{\omega}(v, X) = 0$, since $qX \in H^\perp$ for all $q \in \mathbb{H}$. So $\vec{\omega}'$ is a purely horizontal form. By construction, $\phi^\alpha \wedge \overline{\phi^\alpha}$ is purely vertical/normal. Since

$$0 = \mathrm{d}\vec{\omega} = \mathrm{d}\vec{\omega}' - \mathrm{d}(\phi^\alpha \wedge \overline{\phi^\alpha})$$

we know that $\mathrm{d}\vec{\omega}'$ can have at most one vertical/normal component, and $\mathrm{d}(\phi^\alpha \wedge \overline{\phi^\alpha})$ at most one horizontal component. Hence, by comparing components, we have $\mathrm{d}\vec{\omega}' = 0$ and $\mathrm{d}(\phi^\alpha \wedge \overline{\phi^\alpha}) = 0$.

■

Remark
Notice that the ϕ_0^α gives rise to the connection on $N \longrightarrow M /\!\!/\!\!/\!\!/ G$ generated by the metric when we restrict to N.

Now, a hyperKähler manifold has a canonical 4-form, namely[2]

$$\Omega = \vec{\omega} \wedge\!\!\!\cdot\, \vec{\omega} = \omega_{\mathbb{I}} \wedge \omega_{\mathbb{I}} + \omega_{\mathbb{J}} \wedge \omega_{\mathbb{J}} + \omega_{\mathbb{K}} \wedge \omega_{\mathbb{K}}.$$

In fact, the cohomology class of this 4-form is the 1st Pontryagin class of $\Lambda^2_+ \mathrm{T}^* M$ (see [9]). Our aim is to mimic the construction of the symplectic version of Witten's equivariant integral, but there are hidden dangers in this approach. If we try the approach of taking a complex structure, then not only do we lose the invariance under change of complex structure but we have problems localising the Poincaré dual of $N_{\mathbb{C}} = \mu_{\mathbb{C}}^{-1}(0)$. Other methods involve reducing by a non-compact group which may be fine in the algebraic-geometric sense but the integration is not very well-defined.

Definition 3.3.2 *Let M be a hyperKähler manifold which possesses a tri-Hamiltonian action of the compact Lie group G and $\alpha \in \Omega^\bullet_G(M)$ be an equivariant form. We shall say that α is associated to $\alpha_0 \in \Omega^\bullet(\mathcal{M})$ if*

$$\iota^* \alpha = \pi^* \alpha_0 + \mathrm{d}_{\mathfrak{g}} \beta$$

where $\iota : \vec{\mu}^{-1}(0) \longrightarrow M$ is inclusion and $\pi : \vec{\mu}^{-1}(0) \longrightarrow \mathcal{M}$ is the quotient map and $\beta \in \Omega^\bullet_G\left(\vec{\mu}^{-1}(0)\right)$. In the case that α and α_0 are compactly supported, we shall say that α is compactly associated to α_0 if α is associated to α_0 as above, and the form β is also compactly supported.

[2] We use the dot . to denote the scalar product of quaternions following the motivation from regarding $\Im\mathbb{H}$ as $\mathbb{R}^3$. Hence $\wedge\!\!\!\cdot$ is an operation

$$\wedge\!\!\!\cdot : \Lambda^\bullet(V) \otimes \mathbb{H} \times \Lambda^\bullet(V) \otimes \mathbb{H} \longrightarrow \Lambda^\bullet(V)$$

for any vector space V, given by combining the quaternionic dot product with the wedge product.

Proposition 3.3.3 *The equivariant form*

$$\Omega(y) = (\vec{\omega} + \{\vec{\mu}, y\}) \wedge (\vec{\omega} + \{\vec{\mu}, y\})$$

is an equivariantly closed $\mathbb{R}$*-valued form.*

Proof
For each $q \in \{\mathbb{i}, \mathbb{j}, \mathbb{k}\}$, we have

$$\mathrm{d}_{\mathfrak{g}}(\omega_q + \{\mu_q, \xi\})(\xi) = \{\mathrm{d}\mu_q, \xi\} - X_\xi \lrcorner \omega_q = 0.$$

Hence

$$y \mapsto (\vec{\omega} + \{\vec{\mu}, y\}) \wedge (\vec{\omega} + \{\vec{\mu}, y\}) = \sum_{q \in \{\mathbb{i}, \mathbb{j}, \mathbb{k}\}} (\omega_q + \{\mu_q, y\}) \wedge (\omega_q + \{\mu_q, y\})$$

is an equivariantly closed $\mathbb{R}$-valued form. ∎

Now, there is an issue of compactness arising here. There are no compact hyperKähler manifolds which admit tri-hamiltonian actions of Lie groups, and besides the ADHM construction is a hyperKähler reduction of a non-compact case. However, we can talk about certain submanifolds of non-compact hyperKähler manifolds which reduce under the action of a tri-hamiltonian group action to submanifolds of the quotient.

Definition 3.3.4 *We shall say that the tri-Hamiltonian action of a Lie group G on a hyperKähler manifold M with respects a G-invariant submanifold L of M if for each $p \in L$ any vector normal to L at p lies in* $\ker \mathrm{D}\vec{\mu}_p$.

Remark
For a group action to respect a submanifold L of a hyperKähler manifold, it is necessary that the codimension of L in M should not exceed three times the dimension of the group.

Theorem 3.3.5 *Let M be a hyperKähler manifold that admits a tri-hamiltonian action of the compact Lie group G with moment map $\vec{\mu}$. Let $0 \in \mathfrak{g} \otimes \Im\mathbb{H}$ be a regular value and suppose further that the action of G respects a G-invariant submanifold L , then*

$$\frac{L \cap \vec{\mu}^{-1}(0)}{G}$$

is a submanifold of $M /\!/\!/\!/ G$ with the same codimension as L in M .

Proof
This is an exercise in tranversality. For each $p \in L \cap \vec{\mu}^{-1}(0)$

$$\mathrm{T}_p L + \mathrm{T}_p \vec{\mu}^{-1}(0) \quad = \quad \mathrm{T}_p L + \ker \mathrm{D}\mu_p$$

Since the action respects L, $\ker \mathrm{D}\mu_p$ contains the normal vectors at p, proving that $L \pitchfork \vec{\mu}^{-1}(0)$. Hence this is a G invariant manifold of dimension $n - 3\dim G$ and the result follows. ■

We will prove a Witten-style formula for submanifolds of hyperKähler spaces that are respected by the group action. To do this, we need a number of results.

Proposition 3.3.6 *For $n \in \mathbb{N}$ and $x \in \mathbb{R}\backslash\{0\}$*

$$\lim_{t \longrightarrow \infty} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}+ixy} y^{\frac{n}{2}} \, dy = 0$$

Proof

A calculation.

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+ixy} y^{\frac{n}{2}} &= e^{-tx^2} \int_{\mathbb{R}} e^{-\left(\frac{y}{2\sqrt{t}}-ix\sqrt{t}\right)^2} y^{\frac{n}{2}} \, dy \\
&= 2e^{-tx^2}\sqrt{t} \int_{\mathbb{R}-ix\sqrt{t}} e^{-z^2} (2\sqrt{t}z + 2ixt)^{\frac{n}{2}} \, dz \\
&= 2e^{-tx^2} t^{\frac{n+1}{2}} \int_{\mathbb{R}-i\mathrm{sign}(x)} e^{-z^2} \left(2\frac{z}{\sqrt{t}} + 2ix\right)^{\frac{n}{2}} dz \\
&\quad \text{since either branch of the square root} \\
&\quad \text{is holomorphic away from } 0 \\
&\longrightarrow 0
\end{aligned}$$

as $t \longrightarrow \infty$ provided $x \neq 0$. ■

Theorem 3.3.7 *Let $(M, \vec{\omega})$ be a hyperKähler manifold with a tri-hamiltonian*[3] *action of S^1 and associated hyperKähler moment map $\vec{\mu} : M \longrightarrow \Im\mathbb{H}$. Suppose that 0 is a regular value for $\vec{\mu}$ and let $\mathcal{M} = M/\!\!/\!\!/\!\!/S^1$ be the hyperKähler reduction of M with hyperKähler structure $\vec{\omega}_0$. If $\eta_0 \in \Omega^\bullet(\mathcal{M})$ is compactly supported and compactly associated with $\eta \in \Omega^\bullet_{S^1}(M)$, then we have*

$$\begin{aligned}
&\int_{\mathcal{M}} e^{i\vec{\omega}_0 \wedge \vec{\omega}_0} \left(4i\vec{\omega}_0 \wedge \vec{\omega}_0 + 1\right) \eta_0 \\
= \ & \frac{1}{6\pi i\sqrt{2}} \oint_M e^{i\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y} \eta(\sqrt{y}).
\end{aligned}$$

Proof

Notice that

$$e^{2i\sqrt{y}\vec{\mu}.\vec{\omega}} \eta(\sqrt{y})$$

is a polynomial in $\sqrt{y}$ due to the presence of a (usual) form in the exponent, so we can apply Proposition 3.3.6 to show that the integrand is supported on N.

[3] therefore implying M is not compact

Also, by Lemma 3.3.1 , we have in a neighbourhood of N

$$\begin{aligned}\vec{\omega} \wedge \vec{\omega} &= \vec{\omega}' \wedge \vec{\omega}' \\ &- \vec{\omega}' \wedge \left(\phi \wedge \overline{\phi}\right) \\ &+ 6\phi_0 \wedge \phi_1 \wedge \phi_2 \wedge \phi_3\end{aligned}$$

and by the remark under Lemma 3.3.1, ϕ_0 is a connection on $N \longrightarrow \mathcal{M}$ modified to take real values and scaled by a factor of $\frac{1}{\sqrt{2}}$. From this it is worth trying to estimate $e^{i\vec{\omega} \wedge \vec{\omega}}$ on the space $N = \vec{\mu}^{-1}(0)$.

Claim 3.3.8 *On a small enough tubular neighbourhood U of N we have*

$$\begin{aligned}&e^{i\vec{\omega} \wedge \vec{\omega}} \\ =\ & e^{i\vec{\omega}' \wedge \vec{\omega}' - i\vec{\omega}' \wedge \left(\phi \wedge \overline{\phi}\right) + 6i\phi_0 \wedge \phi_1 \wedge \phi_2 \wedge \phi_3} \\ =\ & e^{i\vec{\omega}' \wedge \vec{\omega}'}\left(1 - i\vec{\omega}' \wedge \left(\phi \wedge \overline{\phi}\right) - 24\vec{\omega}' \wedge \vec{\omega}' \wedge \phi_0 \wedge \phi_1 \wedge \phi_2 \wedge \phi_3 + 6i\phi_0 \wedge \phi_1 \wedge \phi_2 \wedge \phi_3\right).\end{aligned}$$

Proof
Let us simplify matters by temporarily writing

$$\begin{aligned}A &= i\vec{\omega}' \wedge \vec{\omega}' \\ B &= -i\vec{\omega}' \wedge \left(\phi \wedge \overline{\phi}\right) \\ C &= 6i\phi_0 \wedge \phi_1 \wedge \phi_2 \wedge \phi_3.\end{aligned}$$

It is not hard to prove that A, B and C commute (they are even degree forms), and satisfy

$$\begin{aligned}B^2 &= 4AC, \\ C^2 &= 0, \\ BC &= 0.\end{aligned}$$

Then

$$\begin{aligned}(A+B+C)^n &= \sum_{a=0}^{n}\sum_{b=0}^{a} \frac{n!}{a!(n-a)!}\frac{a!}{b!(a-b)!}A^{n-a}B^{a-b}C^b \\ &= \sum_{a=0}^{n}\sum_{b=0}^{a} \frac{n!}{(n-a)!b!(a-b)!}A^{n-a}B^{a-b}C^b \\ &= \sum_{a=0}^{n} \frac{n!}{(n-a)!a!}A^{n-a}B^a + \sum_{a=1}^{n} \frac{n!}{(n-a)!(a-1)!}A^{n-a}B^{a-1}C \\ &= \sum_{a=0}^{2} \frac{n!}{(n-a)!a!}A^{n-a}B^a + nA^{n-1}C \\ &= A^n + nA^{n-1}B + \frac{1}{2}n(n-1)A^{n-2}B^2 + nA^{n-1}C.\end{aligned}$$

Hence

$$\begin{aligned} e^{A+B+C} &= 1+\sum_{n=1}^{\infty}\frac{(A+B+C)^n}{n!} \\ &= 1+\sum_{n=1}^{\infty}\frac{A^n+nA^{n-1}B+\frac{1}{2}n(n-1)A^{n-2}B^2+nA^{n-1}C.}{n!} \\ &= 1+\sum_{n=1}^{\infty}\frac{A^n}{n!}+\sum_{n=1}^{\infty}\frac{A^{n-1}B}{(n-1)!}+\sum_{n=2}^{\infty}\frac{A^{n-2}B^2}{(n-2)!}+\sum_{n=1}^{\infty}\frac{A^{n-1}C.}{(n-1)!} \\ &= e^A+e^AB+4e^AAC+e^AC \\ &= e^A\left(1+B+4AC+C\right), \end{aligned}$$

and substituting back gives the desired result. ∎

Since the normal bundle of N is trivialised by the nowhere vanishing sections $\vec{q}X_\xi$ for $\vec{q}\in\{\mathbb{i},\mathbb{j},\mathbb{k}\}$, we can write $U=N\times V_\varepsilon$ where V_ε is the ball in $\Im\mathbb{H}$ centred at 0 with radius ϵ and such that if $x=(\hat{x},\vec{z})\in N\times V_\varepsilon$, then in this trivialisation $\vec{\mu}(\hat{x},\vec{z})=\vec{z}$. This follows from the fact that for any local submersion f of manifolds, there are local coordinate charts such that f is locally a projection, see [15]. So we if we set

$$W(\vec{z},\sqrt{y})=e^{2i\sqrt{y}\vec{z}.\vec{\omega}},$$

we can write

$$\begin{aligned} &\oint_M e^{i\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2y}\eta(\sqrt{y}) \\ =& \oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'-i\vec{\omega}'\wedge\left(\phi\wedge\overline{\phi}\right)+6i\phi_0\wedge\phi_1\wedge\phi_2\wedge\phi_3+i|\vec{z}|^2y}W(\vec{z},\sqrt{y})\iota^*\eta(\sqrt{y}) \\ =& \oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'-i\vec{\omega}'\wedge\left(\phi\wedge\overline{\phi}\right)+6i\phi_0\wedge\phi_1\wedge\phi_2\wedge\phi_3+i|\vec{z}|^2y}W(\vec{z},\sqrt{y})\left(\pi^*\eta_0+\mathrm{d}_{\mathfrak{g}}\beta(\sqrt{y})\right) \\ =& \oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'-i\vec{\omega}'\wedge\left(\phi\wedge\overline{\phi}\right)+6i\phi_0\wedge\phi_1\wedge\phi_2\wedge\phi_3+i|\vec{z}|^2y}W(\vec{z},\sqrt{y})\pi^*\eta_0 \end{aligned}$$

using a similar argument in Theorem 3.2.6 and the compactness of the support of β. Using the fact that $d\vec{z} = X_i \lrcorner \vec{\omega} = \phi - \phi_0$, we see that

$$
\begin{aligned}
& \oint_M e^{i\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y}\eta(\sqrt{y}) \\
= & \oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'-i\vec{\omega}'\wedge(\phi\wedge\overline{\phi})+6i\phi_0\wedge dvol(\vec{z})+i|\vec{z}|^2 y} W(\vec{z},\sqrt{y})\pi^*\eta_0 \\
= & \oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'+i|\vec{z}|^2 y} W(\vec{z},\sqrt{y})\pi^*\eta_0 \\
- & i\oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'+i|\vec{z}|^2 y}\vec{\omega}'\wedge(d\vec{z}\wedge d\vec{z}^*)\wedge W(\vec{z},\sqrt{y})\pi^*\eta_0 \\
- & 24\oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'+i|\vec{z}|^2 y}\vec{\omega}'\wedge\vec{\omega}'\wedge\phi_0\wedge dvol(\vec{z})\wedge W(\vec{z},\sqrt{y})\pi^*\eta_0 \\
+ & 6i\oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'+i|\vec{z}|^2 y}\phi_0\wedge dvol(\vec{z})\wedge W(\vec{z},\sqrt{y})\pi^*\eta_0 \\
= & 6i\oint_{N\times V_\epsilon} e^{i\vec{\omega}'\wedge\vec{\omega}'+i|\vec{z}|^2 y}(4i\vec{\omega}'\wedge\vec{\omega}+1)\,\phi_0\wedge dvol(\vec{z})\wedge W(\vec{z},\sqrt{y})\pi^*\eta_0
\end{aligned}
$$

where we have also used the Claim 3.3.8. Now we examine the integrals in $\vec{z}$ and y. The integral has the form

$$\lim_{t\longrightarrow\infty}\int_{V_\epsilon}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+i|\vec{z}|^2 y} W(\vec{z},\sqrt{y})dvol(\vec{z})dy.$$

Let Σ_+ be the upper half unit-hemisphere in $\Im\mathbb{H}$ and write

$$\vec{z} = r\vec{\theta}$$

where $\vec{\theta}\in\Sigma_+$.

$$
\begin{aligned}
& \lim_{t\longrightarrow\infty}\int_{V_\epsilon}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+i|\vec{z}|^2 y} W(\vec{z},\sqrt{y})dvol(\vec{z})dy \\
= & \int_{-\epsilon}^{\epsilon}\int_{\Sigma_+}\int_{-\infty}^{\infty} e^{ir^2 y} W(r\vec{\theta},\sqrt{y})r^2 dy\, dvol(\vec{\theta})dr.
\end{aligned}
$$

Now

$$W(\vec{z},\sqrt{y}) = e^{2i\sqrt{y}\vec{z}.\vec{\omega}},$$

and hence is a polynomial in a term of the form $\sqrt{y}\vec{z}.\vec{a}$ for fixed quaternion $\vec{a}$. It makes sense then to evaluate the integral

$$\int_{-\epsilon}^{\epsilon}\int_{\Sigma_+}\int_{-\infty}^{\infty} e^{ir^2 y}(\sqrt{y}\vec{\theta}.\vec{a})^n r^{n+2} dy\, dvol(\vec{\theta})dr$$

for $n \in \mathbb{N}$. Notice that when n is odd, the integral necessarily vanishes as the integral in r is the integral of an odd function. This leaves us with the case n even. So

$$\begin{aligned}
& \int_{-\varepsilon}^{\varepsilon}\int_{\Sigma_+}\int_{-\infty}^{\infty} e^{ir^2 y}(\sqrt{y}\vec{\theta}.\vec{a})^2 n r^{2n+2}\mathrm{d}y\mathrm{d}vol(\vec{\theta})\mathrm{d}r \\
= & \int_{-\varepsilon}^{\varepsilon}\int_{\Sigma_+}\int_{-\infty}^{\infty} e^{ir^2 y}y^n(\vec{\theta}.\vec{a})^{2n} r^{2n+2}\mathrm{d}y\mathrm{d}vol(\vec{\theta})\mathrm{d}r \\
= & \left(\int_{\Sigma_+}(\vec{\theta}.\vec{a})^{2n}\mathrm{d}vol(\vec{\theta})\right)\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty} e^{ir^2 y}y^n r^{2n+2}\mathrm{d}y\mathrm{d}r \\
= & \ const\int_{-\varepsilon}^{\varepsilon} r^{2n+2}D^n(\delta)(r)\mathrm{d}r \\
= & \ D^n(r\mapsto r^{2n+2})(0) \\
= & \ 0.
\end{aligned}$$

Thus

$$\begin{aligned}
& \lim_{t\longrightarrow\infty}\int_{V_\varepsilon}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+i|\vec{z}|^2 y}W(\vec{z},\sqrt{y})\mathrm{d}vol(\vec{z})\mathrm{d}y \\
= & \lim_{t\longrightarrow\infty}\int_{V_\varepsilon}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+i|\vec{z}|^2 y}W(\vec{z},0)\mathrm{d}vol(\vec{z})\mathrm{d}y \\
= & \lim_{t\longrightarrow\infty}\int_{V_\varepsilon}\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+i|\vec{z}|^2 y}\mathrm{d}vol(\vec{z})\mathrm{d}y \\
= & \ 2\pi\int_{V_\varepsilon}\delta(|\vec{z}|^2)\mathrm{d}vol(\vec{z}) \\
= & \ 2\pi
\end{aligned}$$

and

$$\begin{aligned}
& \oint_M e^{i\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y}\eta(\sqrt{y}) \\
= & \ 6i\frac{2\pi}{vol\mathrm{S}^1}\oint_N e^{i\vec{\omega}'\wedge\vec{\omega}'}\left(4i\vec{\omega}'\wedge\vec{\omega}+1\right)\phi_0\wedge\pi^*\eta_0 \\
= & \ 6i\frac{2\pi}{vol\mathrm{S}^1}\oint_N \pi^*\left(e^{i\vec{\omega_0}\wedge\vec{\omega_0}}\left(4i\vec{\omega_0}\wedge\vec{\omega_0}+1\right)\right)\phi_0\wedge\pi^*\eta_0.
\end{aligned}$$

Now, recall that ϕ_0 is $\frac{1}{\sqrt{2}}$ times the connection 1-form which acts as a fibre volume form. Thus we have

$$\begin{aligned}
& \oint_M e^{i\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y}\eta(\sqrt{y}) \\
= & \ 6i\frac{2\pi}{\sqrt{2}}\oint_{\mathcal{M}} e^{i\vec{\omega_0}\wedge\vec{\omega_0}}\left(4i\vec{\omega_0}\wedge\vec{\omega_0}+1\right)\wedge\eta_0 \\
= & \ 6i\pi\sqrt{2}\oint_{\mathcal{M}} e^{i\vec{\omega_0}\wedge\vec{\omega_0}}\left(4i\vec{\omega_0}\wedge\vec{\omega_0}+1\right)\wedge\eta_0.
\end{aligned}$$

If the right hand side is known to exist, then the above argument may be reversed. ■

Corollary 3.3.9 *If M is a hyperKähler manifold with a tri-Hamiltonian action of* S^1 *and L an invariant submanifold respected by* S^1 *. Further, if*

$$\mathcal{L} = (L \cap \vec{\mu}^{-1}(0))/\mathrm{S}^1$$

and $\eta \in \Omega^\bullet_{\mathrm{S}^1}(L)$ is compactly supported and compactly associated to $\eta_0 \in \Omega^\bullet(\mathcal{L})$, then

$$\begin{aligned}
&\int_{\mathcal{L}} e^{i\iota_0^*(\vec{\omega}_0 \wedge \vec{\omega}_0)} \left(4i\iota_0^*(\vec{\omega}_0 \wedge \vec{\omega}_0) + 1\right) \eta_0 \\
&= \frac{1}{6\pi i\sqrt{2}} \oint_L e^{i\iota^*(\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y)}\eta(\sqrt{y}),
\end{aligned}$$

where $\iota : L \hookrightarrow M$ and $\iota_0 : \mathcal{L} \hookrightarrow \mathcal{M}$ are the inclusions .

Proof

This follows the proof of the above theorem, the main consequence of the action respecting the submanifold is that the Poincaré dual of $\vec{\mu}^{-1}(0)$ in M does not vanish when we restrict it to L. ■

Remark

In these integrals, we have not assumed anything about (equivariant) closure. The integrals work well enough without that assumption. However, in order to use localisation, we *do* need assumptions on equivariant closure.

3.3.2 Localisation of HyperKähler Integrals for $G = \mathrm{S}^1$

At first sight, it appears that a localisation formula would be difficult to apply to the Witten-style equation we derived in Theorem 3.3.7 due to the presence of the square roots. What is most important in this formula is that it doesn't depend on choice of square root! We can exchange $-\sqrt{y}$ for $\sqrt{y}$ in the formula and obtain the same result[4]. Hence we have the result for the circle

Theorem 3.3.10

$$\begin{aligned}
&\int_{\mathcal{M}} e^{i\vec{\omega}_0 \wedge \vec{\omega}_0} \left(4i\vec{\omega}_0 \wedge \vec{\omega}_0 + 1\right) \eta_0 \\
&= \frac{1}{12i\pi\sqrt{2}} \oint_M e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2 y} \left(e^{2i\sqrt{y}\vec{\mu}.\vec{\omega}}\eta(\sqrt{y}) + e^{-2i\sqrt{y}\vec{\mu}.\vec{\omega}}\eta(-\sqrt{y})\right).
\end{aligned}$$

Proof

Follows from Theorem 3.3.7 by summing

$$\int_{\mathcal{M}} e^{i\vec{\omega}_0 \wedge \vec{\omega}_0} \left(4i\vec{\omega}_0 \wedge \vec{\omega}_0 + 1\right) \eta_0 = \frac{1}{6\pi i\sqrt{2}} \oint_M e^{i\vec{\omega}\wedge\vec{\omega}+2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y}\eta(\sqrt{y})$$

[4]Notice that we cannot replace $-y$ for y in the symplectic formula without changing the result. This is due to the fact that by changing the sign in this situation, we change a Fourier transform into an inverse Fourier transform

and

$$\int_{\mathcal{M}} e^{i\vec{\omega}_0\wedge\vec{\omega}_0}\left(4i\vec{\omega}_0\wedge\vec{\omega}_0+1\right)\eta_0 \;=\; \frac{1}{6\pi i\sqrt{2}}\oint_M e^{i\vec{\omega}\wedge\vec{\omega}-2i\sqrt{y}\vec{\mu}.\vec{\omega}+i|\vec{\mu}|^2 y}\eta(-\sqrt{y})$$

to get

$$\begin{aligned}&2\int_{\mathcal{M}} e^{i\vec{\omega}_0\wedge\vec{\omega}_0}\left(4i\vec{\omega}_0\wedge\vec{\omega}_0+1\right)\eta_0\\ =\;& \frac{1}{6\pi i\sqrt{2}}\oint_M e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2 y}\left(e^{2i\sqrt{y}\vec{\mu}.\vec{\omega}}\eta(\sqrt{y})+e^{-2i\sqrt{y}\vec{\mu}.\vec{\omega}}\eta(-\sqrt{y})\right).\end{aligned}$$

■

Now for any polynomial $P\in\mathbb{C}[X]$, we know that the polynomial

$$Q: y\mapsto P(y)+P(-y)$$

is a polynomial consisting only of even powers. Hence $Q(\sqrt{y})$ is a perfectly defined polynomial in y. By defining

$$\begin{aligned}\mathrm{Pr}_{ev}&:\mathbb{C}[X]\longrightarrow\mathbb{C}[X^2]\\ \mathrm{Pr}_{ev}(Q)(X)&=\tfrac{1}{2}\left(Q(X)+Q(-X)\right)\end{aligned}$$

we see that

$$\int_{\mathcal{M}} e^{i\vec{\omega}_0\wedge\vec{\omega}_0}\left(4i\vec{\omega}_0\wedge\vec{\omega}_0+1\right)\eta_0=\frac{1}{6\pi i\sqrt{2}}\oint_M e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2 y}\mathrm{Pr}_{ev}\left(z\mapsto e^{2iz\vec{\mu}.\vec{\omega}}\eta(z)\right)(\sqrt{y}). \tag{3.10}$$

Examining the right hand side of the formula in Theorem 3.3.10, we can see that the integrand in Theorem 3.3.10 is an entire function of y. This allows us to apply the theory that Jeffrey and Kirwan develop in [18] to reduce the study after localising to residue formulæ. After applying this we find that in the case for the circle we have the following localisation formula using a similar proof to that of Lemma 3.2.9.

Theorem 3.3.11 , *Suppose $\eta\in\Omega^\bullet_{S^1}(M)$ is equivariantly closed and associated to the closed form $\eta_0\in\Omega^\bullet(\mathcal{M})$. Let $\iota:(M_0)\hookrightarrow M$ be the inclusion of the fixed set in M and e is the equivariant Euler class of the normal bundle of M_0 in M. Then*

$$\begin{aligned}&\int_{\mathcal{M}} e^{i\vec{\omega}_0\wedge\vec{\omega}_0}\left(4i\vec{\omega}_0\wedge\vec{\omega}_0+1\right)\eta_0\\ =\;&\frac{-1}{6\sqrt{2}\pi\operatorname{vol}S^1}\int_{M_0}\mathrm{Coeff}_{y^{-1}}\left[\iota^* e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2 y}\mathrm{Pr}_{ev}\left(z\mapsto\frac{\iota^* e^{2iz\vec{\mu}.\vec{\omega}}\eta(z)}{e(z)}\right)(\sqrt{y})\right].\end{aligned}$$

Proof
First we examine the right hand side of (3.10). By Theorem 3.1.7, we know that

$$\int_M e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2X^2}e^{2iX\vec{\mu}.\vec{\omega}}\eta(X)=\int_{M_0}\frac{\iota^*\left[e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2X^2}e^{2iX\vec{\mu}.\vec{\omega}}\eta(X)\right]}{e(X)}.$$

So we have

$$\begin{aligned}
&\int_M e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2X^2}\mathrm{Pr}_{ev}\left(z\mapsto e^{2iz\vec{\mu}.\vec{\omega}}\eta(z)\right)(X)\\
=\;&\int_{M_0}\iota^*\left[e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2X^2}\mathrm{Pr}_{ev}\left(z\mapsto\frac{e^{2iz\vec{\mu}.\vec{\omega}}\eta(z)}{e(z)}\right)(X)\right]
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{\mathbb{R}}[\mathrm{d}y]e^{-\frac{y^2}{4t}}\int_M e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2y}\mathrm{Pr}_{ev}\left(z\mapsto e^{2iz\vec{\mu}.\vec{\omega}}\eta(z)\right)(\sqrt{y})\\
=\;&\int_{\mathbb{R}}[\mathrm{d}y]e^{-\frac{y^2}{4t}}\int_{M_0}\iota^*\left[e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2y}\mathrm{Pr}_{ev}\left(z\mapsto\frac{e^{2iz\vec{\mu}.\vec{\omega}}\eta(z)}{e(z)}\right)(\sqrt{y})\right]
\end{aligned}$$

by putting $X=\sqrt{y}$. Since the integrand is a rational function of y, the rest now follows from the same proof as Lemma 3.2.9 ■

Using this we can just extend this idea to a product of circles to get a a Witten-style expression and residue theorem for the torus.

Theorem 3.3.12 *Let M be a hyperKähler manifold with tri-Hamiltonian action of the torus $\mathbb{T}^k$. Let $\mathcal{M}=M////\mathbb{T}^k$ and with hyperKähler form $\vec{\omega}_0$, then for each compactly supported $\eta_0\in\Omega^\bullet(\mathcal{M})$ compactly associated to $\eta\in\Omega^\bullet_G(M)$, we have*

$$\begin{aligned}
&\int_{\mathcal{M}} e^{i\vec{\omega}_0\wedge\vec{\omega}_0}\left(4i\vec{\omega}_0\wedge\vec{\omega}_0+1\right)\eta_0\\
=\;&\left(\frac{1}{6\pi i\sqrt{2}}\right)^k\oint\int_M e^{i\vec{\omega}\wedge\vec{\omega}+i\sum_{n=0}^k|\vec{\mu}_\nu|^2y_\nu}\mathrm{Pr}_{ev}\left(\underline{z}\mapsto e^{2i\sum_{\nu=0}kz\vec{\mu}_\nu.\vec{\omega}}\eta(\underline{z})\right)(\sqrt{\underline{y}})
\end{aligned}$$

where $\sqrt{\underline{y}}=(\sqrt{y_1},\dots,\sqrt{y_k})$.

Proof
We decompose $\mathbb{T}^k$ into the product of circles, and recursively use Theorem 3.3.7. ■

Corollary 3.3.13

$$\begin{aligned}
&\left(-6\pi\sqrt{2}\mathrm{vol}\,\mathrm{S}^1\right)^k\int_{\mathcal{M}} e^{i\vec{\omega}_0\wedge\vec{\omega}_0}\left(4i\vec{\omega}_0\wedge\vec{\omega}_0+1\right)\eta_0\\
=\;&\int_{M_0}\mathrm{Coeff}_{y_1^{-1}\dots y_k^{-1}}\left[\iota^*e^{i\vec{\omega}\wedge\vec{\omega}+i\sum_{\nu=1}^k|\vec{\mu}_\nu|^2y_\nu}\mathrm{Pr}_{ev}\left(\underline{z}\mapsto\frac{\iota^*e^{2i\sum_{\nu=1}^k z_\nu\vec{\mu}_\nu.\vec{\omega}}\eta(\underline{z})}{e(\underline{z})}\right)(\sqrt{\underline{y}})\right]\\
=\;&\int_{M_0}\mathrm{Coeff}_{y_1^{-2}\dots y_k^{-2}}\left[\iota^*e^{i\vec{\omega}\wedge\vec{\omega}+i\sum_{\nu=1}^k|\vec{\mu}_\nu|^2y_\nu^2}\frac{\iota^*e^{2i\sum_{\nu=1}^k y_\nu\vec{\mu}_n u.\vec{\omega}}\eta(\underline{y})}{e(\underline{y})}\right]
\end{aligned}$$

Proof
We decompose $\mathbb{T}^k$ into the product of circles, and repeat Theorem 3.3.11. ∎

Now we'd really like to extend this result to more complicated Lie groups than tori, in order to do this we must follow the strategy of Shaun Martin [22] and relate the integrals of a hyperKähler quotient by G to integrals of a hyperKähler quotient by its maximal torus T.

3.3.3 HyperKähler Quotients and Tori

We closely follow Shaun Martin here. Let G be a compact, connected Lie group with maximal torus T. Let Δ be the set of roots of T with $\Delta^{\pm}$ being the set of $\pm$ve roots. We have the orthogonal projection $p : \mathfrak{g} \longrightarrow \mathfrak{t}$, and we denote the moment map $\vec{\mu} : M \longrightarrow \mathfrak{g} \otimes \Im\mathbb{H}$ by $\vec{\mu}_G$ and $(p \otimes \mathbb{1}) \circ \vec{\mu} : M \longrightarrow \mathfrak{t} \otimes \Im\mathbb{H}$ by $\vec{\mu}_T$. Also set

$$\begin{aligned} N_G &= \vec{\mu}_G^{-1}(0), \\ N_T &= \vec{\mu}_T^{-1}(0), \\ \mathcal{M}_G &= N_G/G, \\ \mathcal{M}_T &= N_T/T. \end{aligned}$$

We suppose that 0 is regular for both $\vec{\mu}_G$ and $\vec{\mu}_T$.

Given $\alpha \in \Delta$ we set V_α to be its associated root space (isomorphic to $\mathbb{C}$) and define the vector bundle

$$L_\alpha = N_T \times_T V_\alpha,$$

over $\mathcal{M}_T$. Define

$$V_{\pm} = \bigoplus_{\alpha \in \Delta^{\pm}} L_\alpha$$

and

$$V = V_+ \oplus V_-.$$

We also have

$$\begin{array}{ccc} N_G/T & \overset{\iota}{\hookrightarrow} & \mathcal{M}_T \\ q \downarrow & & \\ \mathcal{M}_G & & \end{array}.$$

Proposition 3.3.14 *[cf Proposition 1.2 of [22]]*

1. *The vector bundle $V_+ \longrightarrow \mathcal{M}_T$ has a section s, which is transverse to the zero section Z, and such that the zero set of s is N_G/T. It follows that the normal bundle*

$$\mathcal{V}(N_G/T; \mathcal{M}_T) \cong \iota^* V_+ \otimes \Im\mathbb{H}$$

2. *Let* Vert(q) *be the vertical subspace of the fibration* $q : N_G/T \longrightarrow \mathcal{M}_G$, *that is the kernel of* q_*. *Then*

$$\mathrm{Vert}(q) \cong \iota^* V_+$$

This is a simple extension of Martin's results in [22].

Theorem 3.3.15 *Let* $e_+ = e(V_+)$, *then for each compactly supported* $\alpha \in \Omega^\bullet(\mathcal{M}_G)$ *with lift* $\tilde{\alpha} \in \Omega^\bullet(\mathcal{M}_T)$ *(that is* $\iota^*\tilde{\alpha} = q^*\alpha$*)*

$$\int_{\mathcal{M}_G} \alpha = \frac{1}{|W|} \int_{\mathcal{M}_T} \tilde{\alpha} \wedge e_+^4,$$

where W *is the Weyl group of* T *in* G.

Proof

(Adapted from the proof of Theorem B [22])

First note that $\iota^* e_+ = e(\mathrm{Vert(q)})$ by 3.3.14. By arguments given in [22], $q_*\iota^*e_+ = |W|$. Thus

$$\begin{aligned}
\int_{\mathcal{M}_G} \alpha &= \frac{1}{|W|} \int_{\mathcal{M}_G} \alpha \wedge q_*\iota^* e_+ \\
&= \frac{1}{|W|} \int_{N_G/T} q^*\alpha \wedge \iota^* e_+ \\
&= \frac{1}{|W|} \int_{N_G/T} \iota^*\tilde{\alpha} \wedge \iota^* e_+ \\
&= \frac{1}{|W|} \int_{\mathcal{M}_T} \iota_*\iota^*(\tilde{\alpha} \wedge e_+) \\
&= \frac{1}{|W|} \int_{\mathcal{M}_T} \tilde{\alpha} \wedge e_+ \wedge e(V_+ \otimes \Im \mathbb{H}) \\
&= \frac{1}{|W|} \int_{\mathcal{M}_T} \tilde{\alpha} \wedge e_+^4.
\end{aligned}$$

■

Lemma 3.3.16 *The equivariant representative of* $e_+ \in \mathrm{H}^\bullet(\mathcal{M}_T)$ *is given by*

$$w(y) = \prod_{\alpha \in \Delta^+} \alpha(y) \in \mathrm{H}^\bullet_T(M).$$

Proof

Since $\pi_\alpha : L_\alpha = N_T \times_T V_\alpha \longrightarrow \mathcal{M}_T$ is a line bundle, we may form $q^*L_\alpha \longrightarrow N_T$ where $q : N_T \longrightarrow \mathcal{M}_T$ is the quotient map. From the standard theory of principal fibrations, we know that

$$q^* L_\alpha \cong N_T \times L_\alpha.$$

This is not trivial in the equivariant sense, and we may put on the equivariant connection $\mathrm{d} + \alpha$. Hence

$$q^* e(L_\alpha)(\xi) = e(q^* L_\alpha)(\xi) = \alpha(\xi).$$

Since there are no usual forms present, we may conclude that $e(L_\alpha)(\xi) = \alpha(\xi)$.

Hence

$$e(V_+) = \prod_{\alpha \in \Delta_+} e(L_\alpha) = \prod_{\alpha \in \Delta_+} \alpha.$$

■

Finally we see that

Theorem 3.3.17 *If η is compactly associated to η_0, then*

$$\begin{aligned} &\int_{\mathcal{M}} e^{i\vec{\omega}_0 \wedge \vec{\omega}_0} \left(4i\vec{\omega}_0 \wedge \vec{\omega}_0 + 1\right) \eta_0 \\ = \;& \left(\frac{1}{6\pi i\sqrt{2}}\right)^k \frac{1}{|W|} \oint_M e^{i\vec{\omega} \wedge \vec{\omega} + i|\vec{\mu}|^2 y} \mathrm{Pr}_{ev} \left(z \mapsto e^{2iz\vec{\mu}.\vec{\omega}} w(z)^4 \eta(z)\right) (\sqrt{y}) \end{aligned}$$

The consequence of this theorem is that now we can replace the Lie group G by its maximal torus and use 3.3.7 recursively à la Guillemin and Kalkman[14].

3.4 Equivariant Characteristic Classes

3.4.1 General Theory

We wish to find an equivariant representative of the μ-form of a submanifold in $\mathbb{R}^4$. Since this form was built up from a characteristic class it is necessary for us to consider the theory of equivariant characteristic classes.

Let $p : V \longrightarrow M$ be an equivariant G bundle over the G-manifold M. We may form the new bundle $V_G \longrightarrow M_G$ by

$$V_G = EG \times_G V \longrightarrow EG \times_G M = M_G$$

and compare the characteristic classes of V with those of G.

Given an equivariant connection ∇ on V we may form a connection ∇_G on V_G by pulling back by the projection $EG \times M \longrightarrow M$ and pushing down to the quotient. Now if K is the structure group of V, then it is also the structure group of V_G, so given a K-invariant polynomial $P \in (\bigodot \mathfrak{k}^*)^K$ we may form the characteristic classes

$$c^P = P(F(\nabla)) \in \Omega^{\text{even}}(M)$$
$$c^P_G = P(F(\nabla_G)) \in \Omega^{even}_G(M)\,.$$

We should like to see how these are related.

Proposition 3.4.1 (Selby [28] p16)
If $f : M \longrightarrow N$ is G-equivariant between G-manifolds inducing the map

$$f_G : EG \times_G M \longrightarrow EG \times_G N$$

and $p : V \longrightarrow N$ a G-equivariant fibre bundle then

$$f^*_G V_G = (f^* V)_G$$

The proof is easy and can be found in detail in [28] Choose a base point $e \in EG$ and let $\iota : M \longrightarrow EG \times M$ be the inclusion $m \mapsto (e, m)$. So for an equivariant vector bundle $p : V \longrightarrow M$ we have

$$\begin{aligned} V &\cong \iota^*(V \times EG) \\ &\cong \iota^* q^* V_G \end{aligned}$$

where $q : EG \times M \longrightarrow M_G$ is the quotient map.We must be careful here; this is an isomorphism of bundles but is not necessarily equivariant. If we set $r = q\iota : M \longrightarrow M_G$ then $r^* : H^*_G(M) \longrightarrow H^*(M)$, and further it can also be shown that

$$r^* c^P_G = c^P$$

3.4.2 de Rham Theory

We now translate the equivariant theory into de Rham formalism. Recall that we make the identification

$$\Omega^*(M_G) \longrightarrow \Omega^*_G(M) = (\Omega^*(M) \otimes \mathbb{C}[\mathfrak{g}^*])^G$$

we can also make the identification

$$\Omega^*(M_G; V_G) \longrightarrow (\Omega^*(M; V) \otimes \mathbb{C}[\mathfrak{g}^*])^G$$

and call these the equivariant forms on M with values in V. Following [8], given an equivariant connection ∇ on $V \longrightarrow M$, we can form the de Rham version $\nabla_{\mathfrak{g}}$ of the corresponding connection on $V_G \longrightarrow M_G$ by setting

$$(\nabla_{\mathfrak{g}} s)\,(\xi) = \nabla(s(\xi)) - X_\xi \lrcorner\, s(\xi).$$

As mentioned in [8] pp210-211, we are motivated by

$$\mathrm{d}_{\mathfrak{g}}{}^2 s(\xi) + \mathcal{L}_\xi s(\xi) = 0$$

to define the equivariant curvature

$$F_{\mathfrak{g}}(\nabla)s(\xi) = \nabla_{\mathfrak{g}}^2 s(\xi) + \mathcal{L}_\xi^V s(\xi)$$

whence

$$F_{\mathfrak{g}}(\nabla)s(\xi) = F(\nabla)s(\xi) - [\nabla, X_\xi \lrcorner]s(\xi) + \mathcal{L}_\xi s(\xi) \tag{3.11}$$

where $\mathcal{L}_\xi^V$ is the Lie derivative on V induced by the action of the vector field X_ξ. Now, given a K-invariant polynomial P (K being the structure group of V), we may obviously form an equivariant characteristic class

$$c^P(\xi) = P(F_{\mathfrak{g}}(\nabla)\xi \in \Omega_G^{\text{even}}(M)\,.$$

How does this relate to the corresponding characteristic class of V/G on M/G?

If G does not act freely on M, then consider the manifold $M^* = M\backslash M_0$ of points with trivial stabiliser. Hence M^*/G is a manifold.

Lemma 3.4.2 *For a vector bundle $V \longrightarrow M^*$*

$$c_G^P(V) = q^* c^P(V/G) + \mathrm{d}_{\mathfrak{g}}\beta$$

where $q : M \longrightarrow M/G$ is the quotient map.

Proof
Given an equivariant connection ∇ on V, we may proceed as in 2.1.1 to obtain a connection ∇' on $V/G \longrightarrow M^*/G$. In turn $q^*\nabla'$ determines an equivariant and horizontal connection on $V \longrightarrow M^*$ and hence an equivariant de Rham connection $\nabla'_{\mathfrak{g}}$ on $V_G \longrightarrow M_G^*$.

We therefore have two connections on $V_G \longrightarrow M_G^*$ namely $\nabla_{\mathfrak{g}}$ and $\nabla'_{\mathfrak{g}}$. By the usual arguments in the theory of characteristic classes

$$P(F_{\mathfrak{g}}(\nabla')) = P(F_{\mathfrak{g}}(\nabla)) + \mathrm{d}_{\mathfrak{g}}\beta.$$

But $P(F_{\mathfrak{g}}(\nabla') = q^*P(F(\nabla'))$ and defines an equivariantly closed form on M^*. So the result follows. ∎

Corollary 3.4.3 *Any equivariant characteristic class of a bundle over a hyperKähler manifold is associated to the characteristic class of the quotient bundle over the hyperKähler reduction.*

Chapter 4

Equivariant Cohomology and the ADHM Construction

We have most of the theory we need to make the calculations necessary to find the linking theory described by Anselmi. However we shall see that the situation is far from plain (or plane) sailing.

4.1 Applications to the ADHM construction

We wish to apply the results on equivariant integration and localisation for hyperKähler quotients to the ADHM construction. Here it is better to pass to the "complex" version given by

$$(T,P) \in \mathfrak{M}^k_{\mathbb{C}} = \left(iu(k) \oplus \mathbb{C}^k\right) \otimes_{\mathbb{R}} \mathbb{H}$$

which has the moment map

$$\vec{\mu}(T,P) = \Im_{\mathbb{H}}(T^*T + PP^*)$$

where $\Im_{\mathbb{H}}$ is the complexification of the operation of taking the quaternionic imaginary part and $*$ means taking the quaternionic conjugate of the complex adjoint.

The reason for this change of approach is that

$$\widetilde{\mathcal{M}} = \mathfrak{M}^k_{\mathbb{R}} /\!\!/\!\!/ \mathrm{O}(k) \cong \mathfrak{M}^k_{\mathbb{C}} /\!\!/\!\!/ \mathrm{U}(k)$$

and $\mathrm{U}(k)$ is connected and has a simpler Lie algebra structure than $\mathrm{O}(k)$. We may also describe the maximal torus of $\mathrm{U}(k)$ more simply than $\mathrm{O}(k)$ and we will be using this to assist us in our localisation. This does not affect any of our previous results.

4.1.1 The Equivariant Euler Class

Our first priority is to work out the fixed point set of the action of the maximal torus

$$\mathbb{T}^k \subset \mathrm{U}(k).$$

To do this, we take the decomposition

$$\mathbb{T}^k = \mathbb{T}_1 \times \ldots \times \mathbb{T}_k$$

where

$$\mathbb{T}_j = \left\{ \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & e^{i\theta_j} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \middle| \theta_j \in \mathbb{R} \right\}.$$

By finding the fixed set of $\mathbb{T}_i$ and the equivariant Euler class of its normal bundle in $\mathfrak{M}_{\mathbb{C}}^k$ we will be able to apply our inductive formula.

Let $\xi_j \in \mathbb{T}_i$ be the generator

$$\xi_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the vector field X_{ξ_j} on $\mathfrak{M}_{\mathbb{C}}^k$ is given by

$$\begin{aligned} X_{\xi_j}(T,P) &= ([\xi_j, T], \xi_j P) \\ &= \left(i \begin{bmatrix} 0 & -T_1 & 0 \\ T_1^* & 0 & T_4 \\ 0 & -T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ iP_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right), \end{aligned}$$

where

$$T = \begin{pmatrix} T' & T_1 & T_2 \\ T_1^* & T'' & T_4 \\ T_2^* & T_4^* & T''' \end{pmatrix}, \text{ and } P = \begin{pmatrix} P_1 \\ \vdots \\ P_k \end{pmatrix}.$$

Let Sing_j^k consist of the points where X_{ξ_j} vanishes, i.e

$$\mathrm{Sing}_j^k := \left\{ \left(\begin{bmatrix} T' & 0 & T_2 \\ 0 & T'' & 0 \\ T_2^* & 0 & T''' \end{bmatrix}, \begin{bmatrix} P' \\ 0 \\ P'' \end{bmatrix} \right) \in \mathfrak{M}_{\mathbb{C}}^k \right\}.$$

Theorem 4.1.1 *The S^1-equivariant Euler class e of the normal bundle of Sing_j^k in $\mathfrak{M}_{\mathbb{C}}^k$ is given by*

$$e(\lambda) = \left(\frac{\lambda}{2\pi} \right)^{4k}.$$

Proof
Set $\mathcal{V}_j = \mathcal{V}(\mathrm{Sing}_j^k \hookrightarrow \mathfrak{M}_{\mathbb{C}}^k)$, the normal bundle of Sing_j^k in $\mathfrak{M}_{\mathbb{C}}^k$. We notice that Sing_j^k is a vector subspace of $\mathfrak{M}_{\mathbb{C}}^k$, so

$$\begin{aligned} \mathrm{T}_{(T,P)}\mathrm{Sing}_j^k &\cong \mathrm{Sing}_j^k, \\ \mathrm{T}_{(T,P)}\mathfrak{M}_{\mathbb{C}}^k &\cong \mathfrak{M}_{\mathbb{C}}^k, \end{aligned}$$

canonically. So

$$(\mathcal{V}_j)_{(T,P)} = \left(\mathrm{T}_{(T,P)}\mathrm{Sing}_j^k\right)^{\perp} = (\mathrm{Sing}_j^k)^{\perp}.$$

It can be shown that

$$\begin{aligned} (\mathrm{Sing}_j^k)^{\perp} &= \{X_{\xi_j}(T,P) | (T,P) \in \mathfrak{M}_{\mathbb{C}}^k\} \\ &= \left\{ \left(\begin{bmatrix} 0 & T_1 & 0 \\ -T_1^* & 0 & T_4 \\ 0 & -T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ P_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \middle| \begin{array}{lcl} T_1 & \in & \mathbb{C}^{j-1} \otimes_{\mathbb{R}} \mathbb{H}, \\ T_4 & \in & \mathbb{C}^{k-j} \otimes_{\mathbb{R}} \mathbb{H}, \\ P_j & \in & \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \end{array} \right\}. \end{aligned}$$

Since $\mathcal{V}_j$ is an equivariant bundle, and the de-Rham operator is a perfectly good equivariant connection on $\mathcal{V}_j$, we see automatically from (3.11) that the equivariant curvature

$$F_{\mathfrak{g}}(\mathrm{d})(\lambda\xi_j) = \mathcal{L}_{\lambda\xi_j}^{\mathcal{V}_j}.$$

Now,

$$\mathcal{L}_{\lambda\xi_j}^{\mathcal{V}_j} \left(\begin{bmatrix} 0 & T_1 & 0 \\ -T_1^* & 0 & T_4 \\ 0 & -T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ P_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = \lambda i \left(\begin{bmatrix} 0 & -T_1 & 0 \\ T_1^* & 0 & -T_4 \\ 0 & T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ P_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

More simply

$$\mathcal{L}_{\lambda\xi_j}^{\mathcal{V}_j}(T_1, T_4, P_j) = (-i\lambda T_1, -i\lambda T_4, i\lambda P_j).$$

From this we are able to deduce that

$$\begin{aligned} e_j(\lambda) &= \mathrm{Pfaff}\left(\frac{1}{2\pi} F_{\mathfrak{g}}(\lambda\xi_j)\right) \\ &= \left(\frac{-\lambda}{2\pi}\right)^{4(k-1)} \left(\frac{\lambda}{2\pi}\right)^4 \\ &= \left(\frac{\lambda}{2\pi}\right)^{4k}. \end{aligned}$$

■

4.1.2 The Universal 2nd Chern Class

In order to find the Universal equivariant 2nd Chern class, we apply the theory in 3.4.2 to the connection $\widehat{\nabla}$ that we created in Section 2.3. Recall that

$$F(\widehat{\nabla}) = v^* \widehat{\mathrm{d}}\mathfrak{R}^* \wedge \mathfrak{F}\widehat{\mathrm{d}}\mathfrak{R}v.$$

Using this it can be quite easily shown that for some section local section $s \in \Omega^0(\mathfrak{M}^k_{\mathbb{C}}; E)$ and $\xi \in \mathrm{u}(k)$, we have

$$\begin{aligned}
\widehat{\nabla}_{X_\xi} s &= X_\xi \lrcorner v^* \widehat{\mathrm{d}} v s \\
&= v^* \widehat{\mathrm{d}}(vs)(X_\xi) \\
&= v^* \frac{\partial}{\partial t}\bigg|_{t=0} \left(\phi_t^{(\xi)}\right)^* vs \\
&= v^* \left(\frac{\partial}{\partial t}\bigg|_{t=0} \left(\phi_t^{(\xi)}\right)^* v \right) s + \frac{\partial}{\partial t}\bigg|_{t=0} \left(\phi_t^{\xi}\right)^* s \\
&= v^* \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} vs + \mathcal{L}_\xi s.
\end{aligned}$$

Hence

$$\begin{aligned}
F_{\mathfrak{g}}(\widehat{\nabla})(\xi) &= F(\widehat{\nabla}) - v^* \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} v \\
&= v^* \left(\widehat{\mathrm{d}}\mathfrak{R}^* \wedge \mathfrak{F}\widehat{\mathrm{d}}\mathfrak{R} - \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \right) v
\end{aligned}$$

and obviously our representative of the equivariant universal Chern class will be

$$c^k(\xi) = \frac{1}{4\pi^2} \mathrm{tr}\left(F_{\mathfrak{g}}(\widehat{\nabla})(\xi) \wedge F_{\mathfrak{g}}(\widehat{\nabla})(\xi) \right) \in \Omega^4_{\mathbb{T}^k}(\mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}).$$

However, c^k is integrable but not smooth on the space of reducibles. To understand c^k on the space of reducibles, if we look at $c^k_{(x,T,P)}$ for

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ P_1 \end{pmatrix}$$

where $T_0 \in \mathbb{H}$, $(T_1, P_1) \in \mathfrak{M}^{k-1}_{\mathbb{C}}$ and $x \neq T_0$, then it is straightforward to show that

$$c^k_{(x,T,P)}(\xi) = c^{k-1}_{(x,T_1,P_1)}(\xi_1)$$

where ξ_1 is the element of $\mathrm{Lie}\mathbb{T}^{k-1}$ got from ξ by removing the first row and column. But,

$$\int_{\mathbb{R}^4} c^k_{(x,t,p)}(\xi)\mathrm{d}x = k$$

for all irreducible $(t,p) \in \mathfrak{M}^k_{\mathbb{C}}$.

Lemma 4.1.2 *For the above* (T,P)

$$\iota_1^* c^k_{(x,T,P)}(\xi) = c^{k-1}_{(x,T_1,P_1)}(\xi_1) + \delta(x - T_0)\mathrm{d}(x - T_0)_1 \wedge \mathrm{d}(x - T_0)_2 \wedge \mathrm{d}(x - T_0)_3 \wedge \mathrm{d}(x - T_0)_4.$$

where $\iota_1 : \mathrm{Sing}_1^k \hookrightarrow \mathfrak{M}_{\mathbb{C}}^k$ *is inclusion and if*

$$\xi = \begin{pmatrix} \xi_{11} & 0 & & \\ 0 & \xi_{22} & & \\ & & \ddots & \\ & & & \xi_{kk} \end{pmatrix}$$

we let

$$\xi_1 = \begin{pmatrix} \xi_{22} & & \\ & \ddots & \\ & & \xi_{kk} \end{pmatrix}.$$

Proof

Set

$$L(\xi) = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}.$$

The splitting of the matrix $L(\xi)$ is in terms of the splitting of $\mathbb{C}^k \otimes_{\mathbb{R}} \mathbb{H} \oplus \mathbb{C}^2$. We have another splitting to consider here due to separating T into T_0 and T_1. Let

$$L(\xi) = \begin{pmatrix} \xi_{11} & 0 \\ 0 & L_1(\xi_1) \end{pmatrix},$$

where $L_1(\xi_1)$ plays same rôle as L for $\mathfrak{M}_{\mathbb{C}}^{k-1}$.

We have

$$\mathcal{R}_{(x,T,P)} = \begin{pmatrix} (T_0 - x)^* & 0 & 0 \\ 0 & (T_1 - x\mathbb{1})^* & P_1 \end{pmatrix} = \begin{pmatrix} -\tilde{x}_0^* & 0 \\ 0 & \mathcal{R}_1 \end{pmatrix}$$

where $\tilde{x_0} = x - T_0$ and $\mathcal{R}_1 = \mathcal{R}_{(x,T_1,P_1)}$. Now

$$\mathcal{F} = (\mathcal{R}\mathcal{R}^*)^{-1} = \begin{pmatrix} |\tilde{x}_0|^{-2} & 0 \\ 0 & \mathcal{F}_1 \end{pmatrix}$$

where $\mathcal{F}_1 = (\mathcal{R}_1\mathcal{R}_1^*)^{-1}$. Since we have difficulties with this when $x = T_0$, we make a small adjustment depending on a parameter ρ which we will shrink to 0.

Set

$$\mathcal{F}_\rho = \begin{pmatrix} \frac{1}{(\rho^2+|\tilde{x}_0|^2)} & 0 \\ 0 & \mathcal{F}_1 \end{pmatrix}.$$

Define

$$\varpi_\rho = \mathbb{1} - \mathcal{R}^*\mathcal{F}_\rho\mathcal{R} = \begin{pmatrix} \frac{\rho^2}{(\rho^2+|\tilde{x}_0|^2)} & 0 \\ 0 & \varpi_1 \end{pmatrix}$$

where $\varpi_1 = \mathbb{1} - \mathcal{R}_1^*\mathcal{F}_1\mathcal{R}_1$. We note that

$$\iota_1^*\mathrm{d}\mathcal{R} = \begin{pmatrix} -\widehat{\mathrm{d}}\tilde{x}_0^* & 0 \\ 0 & \widehat{\mathrm{d}}\mathcal{R}_1 \end{pmatrix}.$$

So we have

$$
\begin{aligned}
& \iota_1^* c^k_{(x,T,P)}(\xi) \\
= \; & \lim_{\rho \longrightarrow 0} \frac{1}{4\pi^2} \Re \mathrm{tr} \left[\left((\iota_1^* \widehat{\mathrm{d}} \mathcal{R}^* \wedge \mathcal{F}_\rho \iota_1^* \widehat{\mathrm{d}} \mathcal{R} \varpi - L(\xi) \varpi \right)^2 \right] \\
= \; & \lim_{\rho \longrightarrow 0} \frac{1}{4\pi^2} \Re \left(\frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^4} \widehat{\mathrm{d}} \tilde{x}_0 \wedge \widehat{\mathrm{d}} \tilde{x}_0^* \wedge \widehat{\mathrm{d}} \tilde{x}_0 \wedge \widehat{\mathrm{d}} \tilde{x}_0^* \right. \\
- \; & \left. \frac{2\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^3} \widehat{\mathrm{d}} \tilde{x}_0 \wedge \widehat{\mathrm{d}} \tilde{x}_0^* \xi_{11} + \frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)} \xi_{11}^2 \right) \\
+ \; & c^{k-1}_{(x,T_1,P_1)}(\xi_1) \\
= \; & \lim_{\rho \longrightarrow 0} \frac{1}{4\pi^2} \Re \left(\frac{24\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^4} \widehat{\mathrm{d}}(\tilde{x}_0)_1 \wedge \widehat{\mathrm{d}}(\tilde{x}_0)_2 \wedge \widehat{\mathrm{d}}(\tilde{x}_0)_3 \wedge \widehat{\mathrm{d}}(\tilde{x}_0)_4 \right. \\
- \; & \left. \frac{2\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^3} \widehat{\mathrm{d}} \tilde{x}_0 \wedge \widehat{\mathrm{d}} \tilde{x}_0^* \xi_{11} + \frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)} \xi_{11}^2 \right) \\
+ \; & c^{k-1}_{(x,T_1,P_1)}(\xi_1).
\end{aligned}
$$

Now the terms of the form

$$\frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^n}$$

are distributional in the limit as $\rho_{ii} \longrightarrow 0$, so it is worth integrating them against a compactly supported test function $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ that is, calculating

$$\lim_{\rho \longrightarrow 0} \int_{\mathbb{R}^4} \frac{\rho^4}{(\rho^2 + |y|^2)^n} f(y) \mathrm{d}y.$$

It isn't hard to show that as distributions

$$\lim_{\rho \longrightarrow 0} \frac{\rho^4}{(\rho^2 + |y|^2)^4} = \frac{\pi^2}{6} \delta(y),$$

$$\lim_{\rho \longrightarrow 0} \frac{\rho^4}{(\rho^2 + |y|^2)^3} = 0$$

and after a little work is is possible to show that

$$\lim_{\rho \longrightarrow 0} \frac{\rho^4}{(\rho^2 + |y|^2)^2} = 0$$

as a distribution. We are therefore left with the result that

$$\iota_1^* c^k_{(x,T,P)}(\xi) = c^{k-1}_{(x,T_1,P_1)}(\xi_1) + \delta(x - T_0) \mathrm{d}(x - T_0)_1 \wedge \mathrm{d}(x - T_0)_2 \wedge \mathrm{d}(x - T_0)_3 \wedge \mathrm{d}(x - T_0)_4.$$

as stated. ∎

For a submanifold $\Sigma \subset \mathbb{R}^4$, define

$$\mu^k(\Sigma)_{(T,P)}(\xi) = \int_{x \in \Sigma} c^k_{(x,T,P)}(\xi) = \int_{x \in \mathbb{R}^4} c^k_{(x,T,P)}(\xi) \wedge PD(\Sigma)_x.$$

Corollary 4.1.3 *With T, P, ι_1 as above, we have*

$$\iota_1^* \mu^k(\Sigma)_{(x,T,P)}(\xi) = \mu^{k-l}(\Sigma)_{(T_1,P_1)}(\xi_1) + \mathrm{PD}(\Sigma)_{T_0}.$$

Corollary 4.1.4 *If ι is the inclusion of the set of points of $\mathfrak{M}_{\mathbb{C}}^k$ fixed under $\mathbb{T}^k$, and $p_i : \mathfrak{M}_{\mathbb{C}}^k \longrightarrow \mathbb{R}^4$ is the projection $T \mapsto T_{ii}$ then*

$$\iota^* \mu^k(\Sigma)(\xi) = \sum_{i=1}^{k} p_i^* \mathrm{PD}(\Sigma).$$

Corollary 4.1.5 *If $\mu^k(\Sigma) \in \Omega^{4-\dim\sigma}(\frac{\mathfrak{M}_{\mathbb{C}}^k}{\mathrm{U}(k)\mathrm{Sp}(1)})$ then for (T, P) and ι_1 as above we have*

$$\iota_1^* \pi^* p^* \mu^k(\Sigma)_{(T,P)} = \pi^* p^* \mu^{k-1}(\Sigma)_{(T_1,P_1)} + p_1^* \mathrm{PD}(\Sigma)_{T_0} + \pi^* p^* \widehat{\mathrm{d}}\gamma(\Sigma).$$

for some $\gamma(\Sigma) \in \Omega^{3-\dim\Sigma}(\frac{\mathfrak{M}_{\mathbb{C}}^k}{\mathrm{U}(k)\mathrm{Sp}(1)})$, and p_1 the projection as in Corollary 4.1.4.

Proof

Now, we have a chain of quotients

$$\begin{array}{ccccc} \mathfrak{M}_{\mathbb{C}}^k & \xrightarrow{\pi} & \frac{\mathfrak{M}_{\mathbb{C}}^k}{\mathrm{U}(k)} & \xrightarrow{p} & \frac{\mathfrak{M}_{\mathbb{C}}^k}{\mathrm{U}(k)\mathrm{Sp}(1)} \\ & & \cup & & \cup \\ & & \widetilde{\mathcal{M}}_k & & \mathcal{M}_k \end{array}.$$

So the form $\pi^* p^* \mu^k(\Sigma)$ is now a closed, $\mathrm{U}(k)$-basic $4-\dim\Sigma$ degree form on $\mathfrak{M}_{\mathbb{C}}^k$. Since $\mu^k(\Sigma)(\xi) \in \Omega_{\mathrm{U}(k)}(\mathfrak{M}_{\mathbb{C}}^k)$ was formed using equivariant Chern-Weil theory and by Lemma 3.4.2, we have

$$\mu^k(\Sigma)(\xi) = \pi^* p^* \mu^k(\Sigma) + \int_\Sigma \mathrm{d}_{\mathfrak{g}} \beta^k(\xi).$$

Also by Corollary 4.1.3 we have

$$\iota_1^* \mu^k(\Sigma)_{(x,T,P)}(\xi) = \mu^{k-l}(\Sigma)_{(T_1,P_1)}(\xi_1) + \mathrm{PD}(\Sigma)_{T_0}.$$

Thus

$$\iota_1^* \pi^* p^* \mu^k(\Sigma) = \pi^* p^* \mu^{k-1}(\Sigma)_{(T_1,P_1)} + \mathrm{PD}(\Sigma)_{T_0} + \int_\Sigma \mathrm{d}_{\mathfrak{g}} \left(\beta^{k-1} - \iota_1^* \beta^k\right)(\xi).$$

The left hand side is independent of ξ, so $\mathrm{d}_{\mathfrak{g}} \left(\beta^{k-1} - \iota_1^* \beta^k\right)(\xi)$ is an exact, $\mathrm{U}(k)$- basic form, thus the right hand side is in the same de Rham cohomology class as the left. The result follows since $\pi^* p^* \mu^k(\Sigma)$, $\pi^* p^* \mu^{k-1}(\Sigma)$ and the Poincaré dual are $\mathrm{U}(k)\mathrm{Sp}(1)$-basic. By construction, $\iota_1^* \mathrm{d}_{\mathfrak{g}} \beta^k$ does not depend on T_0 and thus agrees with $\mathrm{d}_{\mathfrak{g}} \beta^{k-1}$ giving the result. ■

4.1.3 Integrability of the Donaldson μ map

What is not altogether clear is that the form representing the Donaldson polynomial is actually integrable. Indeed there are various technicalities in forming the these polynomials that Donaldson and Kronheimer discuss in Chapter 9 of [13]. Our approach will be from a functional analytic viewpoint.

Definition 4.1.6 *Given compact submanifolds $\Sigma_1, \ldots \Sigma_l$ of $\mathbb{R}^4$ we define the Donaldson functional on compactly supported functions of $\frac{\mathfrak{M}^k_{\mathbb{C}}}{\mathrm{U}(k)\mathrm{Sp}(1)}$ by*

$$\mathrm{Don}_k(\Sigma_1, \ldots, \Sigma_l)(\phi) = \int_{\mathcal{M}_k} \phi\, \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l).$$

This is certainly well defined, the above argument shows that the representatives of the μ classes on the reducible space are distributional in nature and integrable, and thus the integral exists for any compactly supported function ϕ.

4.1.4 Computing the integrals

We must, however, make a slight alteration to the situation since the action of $\mathrm{U}(k)$ is not free on $\vec{\mu}^{-1}(0)$. Instead we choose $\vec{\zeta}_0 \in \Im\mathbb{H}$ and change the moment map to

$$\vec{\mu}(T, P) = \Im(T^*T + PP^*) - \vec{\zeta}_0\mathbb{1}.$$

We have to decide on how best to approach the integration.

Let $\Sigma_1, \ldots, \Sigma_l$ be compact submanifolds of $\mathbb{R}^4$ of dimensions $d_1, \ldots, d_l$ respectively such that

$$\sum_{i=1}^{l}(4 - d_i) = 8k - 3,$$

that is

$$\sum_{i=1}^{l} d_i = 4l - 8k + 3.$$

Then $\alpha = \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l)$ is represented by a form of top degree on $\mathcal{M}_k$.

Now the de-Rham operator $\widehat{\mathrm{d}}$ splits on $\mathbb{R}^4 \times \mathcal{M}_k$

$$\widehat{\mathrm{d}} = \mathrm{d} + \delta$$

where δ is the de Rham differential on $\mathcal{M}$. Since both $\mathrm{U}(k)$ and $\mathrm{Sp}(1)$ act trivially on $\mathbb{R}^4$, we see that

$$\mathrm{d}_{\mathfrak{g}}\eta(\gamma) = \mathrm{d}\eta(\gamma) + \delta\eta(\gamma) - X_\gamma \lrcorner \eta(\gamma) = \widehat{\mathrm{d}}\eta(\gamma) - X_\gamma \lrcorner \eta(\gamma),$$

for any $\eta \in \Omega_G^\bullet\left(\mathbb{R}^4 \times \mathfrak{M}_{\mathbb{C}}^k\right)$, where following the notation in Chapter 2, we reserve d for the de-Rham differential on $\mathbb{R}^4$ and δ the differential on $\mathfrak{M}_{\mathbb{C}}^k$ and the total differential $\widehat{\mathrm{d}} = \mathrm{d} + \delta$. Now suppose without loss of generality that Σ_1 is not a point, and that Ξ_1 is a Seifert surface spanning Σ_1, i.e $\partial \Xi_1 = \Sigma_1$. Then

$$
\begin{aligned}
0 &= \int_{\Xi_1} \widehat{\mathrm{d}} c^k \\
&= \int_{\Xi_1} \mathrm{d} c^k + \int_{\Xi_1} \delta c^k \\
&= \int_{\Sigma_1} c^k + \delta \int_{\Xi_1} c^k
\end{aligned}
$$

i.e

$$
\int_{\Sigma_1} c^k = -\delta \int_{\Xi_1} c^k. \tag{4.1}
$$

Hence we may take

$$
\beta = -\int_{\Xi_1} c^k \wedge \int_{\Sigma_2} c^k \wedge \ldots \wedge \int_{\Sigma_l} c^k
$$

and

$$
\alpha = \delta \beta.
$$

So as we saw above,

$$
\begin{aligned}
& \operatorname{Don}_k(\Sigma_1, \ldots, \Sigma_l)(\phi) \\
= & \int_{\mathcal{M}_k} \phi\, \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) \\
= & \int_{\widetilde{\mathcal{M}}_k} p^*\phi\, p^* \widehat{\mathrm{d}} \left(\mu(\Xi_1) \wedge \ldots \wedge \mu(\Sigma_l)\right) \wedge \Theta \\
= & \int_{\widetilde{\mathcal{M}}_k} e^{i\vec{\omega_0} \wedge \vec{\omega_0}} \left(4 i \vec{\omega_0} \wedge \vec{\omega_0} + 1\right) p^*\phi\, p^* \mathrm{d} \left(\mu(\Xi_1) \wedge \ldots \wedge \mu(\Sigma_l)\right) \wedge \Theta
\end{aligned}
$$

since

$$
p^*\phi p^* \mathrm{d} \left(\mu(\Xi_1) \wedge \ldots \wedge \mu(\Sigma_l)\right) \wedge \Theta
$$

already has maximal degree. This form is associated to

$$
\eta = \pi^* \left(p^*\phi p^* \widehat{\mathrm{d}} \left(\mu(\Xi_1) \wedge \ldots \wedge \mu(\Sigma_l)\right) \wedge \Theta\right) \in \Omega_{\mathrm{U}(k)}^{8k}\left(\mathfrak{M}_{\mathbb{C}}^k\right),
$$

which is basic and de Rham closed by construction and compactly supported.

We can now use Theorem 3.3.17 applied to this form.

$$
\begin{aligned}
& \operatorname{Don}_k(\Sigma_1, \ldots, \Sigma_l)(\phi) \\
= & \left(\frac{1}{6\pi i \sqrt{2}}\right)^k \frac{1}{|S_k|} \oint_{\mathfrak{M}_{\mathbb{C}}^k} e^{i\vec{\omega} \wedge \vec{\omega} + i|\vec{\mu}|^2 y} \operatorname{Pr}_{ev}\left(z \mapsto e^{2iz\vec{\mu}.\vec{\omega}} w(z)^4 \eta\right)(\sqrt{y}), \\
= & \left(\frac{1}{6\pi i \sqrt{2}}\right)^k \frac{1}{k!} \oint_{\mathfrak{M}_{\mathbb{C}}^k} e^{i\vec{\omega} \wedge \vec{\omega} + i|\vec{\mu}|^2 y} \operatorname{Pr}_{ev}\left(z \mapsto e^{2iz\vec{\mu}.\vec{\omega}} w(z)^4\right)(\sqrt{y}) \eta,
\end{aligned}
$$

and use the localisation theorem to prove

Theorem 4.1.7

$$\mathrm{Don}_k(\Sigma_1, \dots, \Sigma_l)(\phi) = \lambda(|\vec{\zeta}_0|)\mathcal{P}(\Sigma_1, \dots, \Sigma_l)(\phi)$$

for λ a suitable polynomial, and $\mathcal{P}$ a topological object depending on the arrangements of the Σ_i in $\mathbb{R}^4$ and upon the test function ϕ .

Proof

We localise the integral with respect to the $(k-1)$-torus in stead of the k-torus since there is a problem with the form Θ at $|P| = 0$ which is the fixed set of the full k-torus . We hope to be able to express the integral then in terms of the Donaldson polynomials for charge $k = 1$. We are not interested in the constant multiples that occur here, so they will be largely forgotten.

$$\begin{aligned}
&\mathrm{Don}_k(\Sigma_1, \dots, \Sigma_l)(\phi) \\
= \;& \left(\frac{1}{6\pi i\sqrt{2}}\right)^k \frac{1}{k!} \oint_{\mathfrak{M}_{\mathbb{C}}^k} e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}|^2 y} \mathrm{Pr}_{ev}\left(z \mapsto e^{2iz\vec{\mu}.\vec{\omega}} w(z)^4\right)(\sqrt{y})\eta, \\
= \;& const \oint_{(\mathbb{R}^4)^{k-1}\times\mathfrak{M}_{\mathbb{C}}^1} \mathrm{Coeff}_{y_1^{-2}\cdots y_{k-1}^{-2}} \left[\frac{\iota^* e^{i\vec{\omega}\wedge\vec{\omega}+i\sum_{\nu=1}^{k}\left(|\vec{\zeta}_0|^2 y_\nu^2 + 2y_\nu\vec{\zeta}_0.\vec{\omega}\right)} w(y)^4}{y_1^{4k} y_2^{4k-4}\cdots y_{k-1}^{8}}\right] \iota^*\eta.
\end{aligned}$$

Now restricted to the fixed set of the $(k-1)$-torus we have by Corollary 4.1.5

$$\iota^*\eta \;=\; \iota^* \prod_{j=1}^{l} \left(\pi^* p^* \mu^1(\Sigma_j) + \sum_{m=1}^{k-1} p_i^* PD(\Sigma_j)\right) \wedge \iota^*\pi^*\Theta$$

where p_i are the projections described in Corollary 4.1.4. Thus the form of the Donaldson polynomial can be seen certainly as the product of a polynomial in $|\vec{\zeta}_0|$ and a sum of integrals of the form

$$\begin{aligned}
&\int_{\mathfrak{M}_{\mathbb{C}}^1\times(\mathbb{R}^4)^{k-1}} (\iota^*\pi^*p^*\phi)\; \pi^*\left(\Theta \wedge \prod_{i\in I} p^*\mu^1(\Sigma_i)\right) \wedge \prod_B \prod_{b\in B} \mathrm{PD}(\Sigma_b) \\
= \;& \int_{\mathcal{M}_1\times(\mathbb{R}^4)^{k-1}} (\iota^*\phi) \left(\prod_{i\in I} \mu^1(\Sigma_i))\right) \wedge \prod_B \prod_{b\in B} \mathrm{PD}(\Sigma_b)
\end{aligned}$$

where I is a subset of $\{1, \dots, l\}$, and B runs over all subsets that form a partition of $\{1, \dots, l\}\backslash I$.
■

4.1.5 $k = 1$ Revisited from the Topological Viewpoint

We present a slightly different approach to the theory of $k = 1$. Here we use the Poincaré duality property detailed in Donaldson's paper [12]. We state it in the version it appears in [13].

Lemma 4.1.8 (Corollary 5.3.3 of [13] p199) *Let X be a simply connected Riemannian 4-manifold and $E \longrightarrow X$ have $c_2(E) = 1$, and let $\tau : X \longrightarrow \mathcal{B}^*_{X,E}$ be any map into the space of the gauge equivalence classes of irreducible connections on E with the property that for all x, the connection $\tau(x)$ is flat and trivial outside some ball of finite diameter centred on x. Then the composite*

$$\mathrm{H}_2(X;\mathbb{Z}) \xrightarrow{\mu} \mathrm{H}^2(\mathcal{B}^*_{X,E}) \xrightarrow{\tau^*} \mathrm{H}^2(X;\mathbb{Z})$$

is the Poincaré duality isomorphism.

Definition 4.1.9 *We will call such a τ a tractator*

We can prove this at the level of forms to show that in the case of $X = \mathrm{S}^4$ we have the following

Lemma 4.1.10 *For a tractator $\tau : \mathrm{S}^4 \longrightarrow \mathcal{B}^*_{\mathrm{S}^4,E}$, we have for each submanifold Σ of S^4*

$$\int_\Sigma \iota^*_\Sigma \alpha = \int_{\mathrm{S}^4} \alpha \wedge \tau^* \mu(\Sigma)$$

for any $\alpha \in \Omega^{\dim \Sigma}(\mathrm{S}^4)$, where ι_Σ is the inclusion of Σ in S^4.

Now, for $\varepsilon > 0$ let

$$\mathfrak{M}^1_{\mathbb{C}\varepsilon} = \{ (T,P) \in \mathfrak{M}^1_{\mathbb{C}} \mid |P| \geq \varepsilon \}$$

and

$$\mathcal{M}_\varepsilon = (\mathfrak{M}^1_{\mathbb{C}\varepsilon} /\!/\!/\!/ \mathrm{U}(1))/\mathrm{Sp}(1).$$

This is a manifold with boundary.

Let $\Sigma_1, \ldots, \Sigma_l$ be submanifolds of S^4 with dimensions $d_1, \ldots, d_l$ respectively such that

$$d_1 + \ldots + d_l = 4l - 5.$$

Suppose w.l.o.g that Σ_1 is not a point and let Ξ_1 be a Seifert manifold for it. then from above we know that

$$\mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) = \mathrm{d}\left(\mu(\Xi_1) \wedge \mu(\Sigma_2) \wedge \ldots \wedge \mu(\Sigma_l)\right).$$

Hence

$$\begin{aligned}
\int_{\mathcal{M}} \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) &= \lim_{\varepsilon \longrightarrow 0} \int_{\mathcal{M}_\varepsilon} \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) \\
&= \lim_{\varepsilon \longrightarrow 0} \int_{\partial \mathcal{M}_\varepsilon} \mu(\Xi_1) \wedge \mu(\Sigma_2) \wedge \ldots \wedge \mu(\Sigma_l) \\
&= \int_{S^4} \tau^* \mu(\Xi_1) \wedge \tau^* \mu(\Sigma_2) \wedge \ldots \wedge \tau^* \mu(\Sigma_l) \\
&\quad \text{for some appropriate tractator } \tau. \\
&= \int_{S^4} \mathrm{PD}(\Xi_1) \wedge \mathrm{PD}(\Sigma_2) \wedge \ldots \wedge \mathrm{PD}(\Sigma_l) \\
&= \text{intersection number of } \bigcap_{i=2}^{l} \Sigma_i \text{ with } \Xi_1 \\
&= \text{linking number of } \Sigma_1 \text{ and } \bigcap_{i=2}^{l} \Sigma_i
\end{aligned}$$

The possible configurations for Donaldson numbers in the case $k = 1$ depend on the relation

$$\sum_{j=1}^{l} d_i = 4l - 8k + 3 = 4l - 5$$

where d_i is the dimension of the submanifold Σ_i. When $l = 2$, we know that $d_1 + d_2 = 3$, so the only possible configurations are

d_1	d_2
0	3
1	2

.

4.1.6 Is there any linking for $k \geq 2$?

It would be prudent to examine $\mathcal{P}(\Sigma_1, \ldots, \Sigma_l)$. Recall that

$$\begin{aligned}
&\mathcal{P}(\Sigma_1, \ldots, \Sigma_l)(\phi) \\
= &\int_{\mathbb{H}^{k-1} \times \mathcal{M}_1} \iota^* \phi \left(\mu^1(\Sigma_1) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_1) \right) \wedge \ldots \wedge \left(\mu^1(\Sigma_l) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_l) \right)
\end{aligned}$$

and let us look at the various configurations. Choose submanifolds $\Sigma_1, \ldots \Sigma_l$ of $\mathbb{R}^4$ of dimensions $d_1, \ldots, d_l$ respectively. Then

$$\sum_{i=1}^{l} (4 - \mathrm{d}_i) = 8k - 3 \tag{4.2}$$

yields

$$\sum_{i=1}^{l} d_i = 4l - 8k + 3. \tag{4.3}$$

Definition 4.1.11 *Let K be a finite set. Then*

1. *we call an n-tuple I of elements of K an ordered subset of K whenever $I = (i_1, \dots, i_p)$ we have $i_j \neq i_r$ for all $j \neq r$;*

2. *we say $n \in I = (i_1, \dots, i_p)$ if there is j such that $i_j = n$;*

3. *we shall write $\#I = p$ if $I = (i_1, \dots i_p)$;*

4. *we shall say that a collection $I_1, \dots I_m$ of ordered subsets of K is a partition of K if for each $n \in K$ there is precisely one r such that $n \in I_r$.*

Now as we said earlier, each term in $\mathcal{P}$ is $\pm \mathrm{Don}_1(\Sigma_{I_1}) \prod_{j>1}^{p} \sharp(\Sigma_{I_i})$ for ordered subsets $I_1, \dots, I_p$ that partition $\{1, \dots, l\}$. For this to give a nonzero contribution to $\mathcal{P}$ we need I_1 to have at least 2 elements and each of the other I_i at least 1 element. Now I_1 must satisfy

$$\sum_{j \in I_1} (4 - \mathrm{d}_j) = 8(1) - 3 = 5 \tag{4.4}$$

that is

$$\sum_{j \in I_1} d_j = 4\#I_1 - 5. \tag{4.5}$$

Now we know that for each of $I_j, j > 1$ we need $\sharp(\Sigma_{I_j}) \neq 0$ so we must have for each $j > 2$,

$$\sum_{q \in I_j} (4 - d_q) = 4$$

that is

$$\sum_{q \in I_j} (d_q) = 4\#I_j - 4.$$

Now

$$\begin{aligned}
\sum_{j=1}^{l} d_j &= \sum_{n=1}^{p} \sum_{j \in I_n} d_j \\
&= \sum_{j \in I_1} d_j + \sum_{n=2}^{p} \sum_{j \in I_n} d_j \\
&= 4\#I_1 - 5 + \sum_{n=2}^{p} (4\#I_n - 4) \\
&= 4 \sum_{n=1}^{p} \#I_n - 5 - 4(p-1) \\
&= 4l - 4p - 1.
\end{aligned}$$

Thus we must have

$$4l - 4p - 1 = 4l - 8k + 3$$

i.e.

$$p = 2k - 1.$$

So k controls the number of ordered subsets of $\{1, \dots, l\}$ that form a partition, moreover this number has to be odd.

However, if we examine the form of $\mathcal{P}$ more closely,

$$\begin{aligned} &\mathcal{P}(\Sigma_1, \dots, \Sigma_l)(\phi) \\ = &\int_{\mathbb{H}^{k-1} \times \mathcal{M}_1} \iota^* \phi \left(\mu^1(\Sigma_1) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_1) \right) \wedge \dots \wedge \left(\mu^1(\Sigma_l) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_j) \right), \end{aligned}$$

we see that each term is the product of sums of k terms, so for any ordered subsets $I_1, \dots, I_p$ that partition $\{1, \dots, l\}$ and give non-zero contribution to $\mathcal{P}$ must satisfy

$$p \leq k.$$

Thus we have

$$k \geq p = 2k - 1$$

which is impossible for $k \in \mathbb{N}$. Hence $\mathcal{P}$ is a trivial topological number, assigning 0 to any set of submanifolds of $\mathbb{R}^4$ and any test function ϕ. We have therefore proved

Theorem 4.1.12 *For $k \geq 2$, and any compactly supported ϕ there are no anomalies, i.e for $k \geq 2$*

$$\mathrm{Don}_k(\Sigma_1, \dots, \Sigma_l)(\phi) = 0$$

for all submanifolds $\Sigma_1, \dots, \Sigma_l$.

We proved this for the resolution of the moduli space. Since the integral is identically 0 on the resolution, it must also be zero on the moduli space itself, thus agreeing with the infinite dimensional construction.

4.1.7 Concluding Remarks

Although we have had something of a disappointment that there is no linking number for $k > 1$ on the moduli space of instantons, nor on any resolutions, we have developed some potentially powerful techniques in finding formulæ for the cohomology of a hyperKähler reduction. One can hope that the technique for hyperKähler manifolds with boundaries may produce information about the topology of the higher instanton spaces by looking at the topology of the end and concluding that the Moduli space is a cone on this manifold. Also there may be something to be said about the perturbed moduli spaces with their relationship with the Seiberg-Witten equations. The techniques introduced here may also provide valuable information about the non-Abelian Seiberg-Witten equations if we can compactify the gauge group.

The techniques used here, I am sure, would be useful in proving similar results Quaternionic Kähler reductions, though it might be difficult to get Theorem 3.3.1 to work due to the non-integrability of the complex structures. However, a similar result might be proved by looking at the subbundle generated by the vector fields $\vec{q}X_\xi$. It may even be possible to produce similar results for Sasakian and 3-Sasakian manifolds whose foliation is induced by a group action.

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